1. EULER ANGLES

10/20/11

$$p' = R_z(\gamma) R_y(\beta) R_x(\alpha) p$$

TO ANIMATE

$$M_1 = R_z\left(\frac{\gamma}{100}\right) R_y\left(\frac{\beta}{100}\right) R_x\left(\frac{\alpha}{100}\right)$$

$$M_2 M_1 M_1 p$$

RESULT LOOKS BAD

"GIMBAL LOCK"

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2. MATRIX 3×3

3. VECTOR
AXIS a $(1,1,0) \rightarrow (.7,.7,0)$

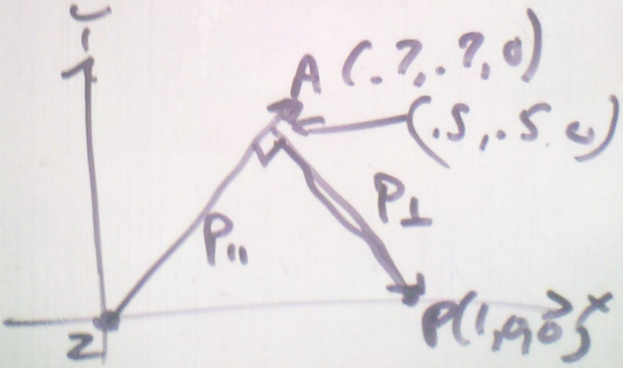
2. MATRIX 3×3

3. VECTOR

AXIS a $(1,1,0) \to (.7,.7,0)$

POINT $P\,(1,0,0)$

$\Theta = 90^\circ$ $\cos\theta = 0$ $\sin\theta = 1.$



$$P_{\parallel} = \underset{.7}{(a \cdot p)}\, a = ([illegible]) \quad (.5, .5, 0)$$

$P_{\parallel} = (a\cdot p)a = ($[illegible]$)$

.?

$(.5,.5,0)$

$P_{\perp} = P - P_{\parallel}$

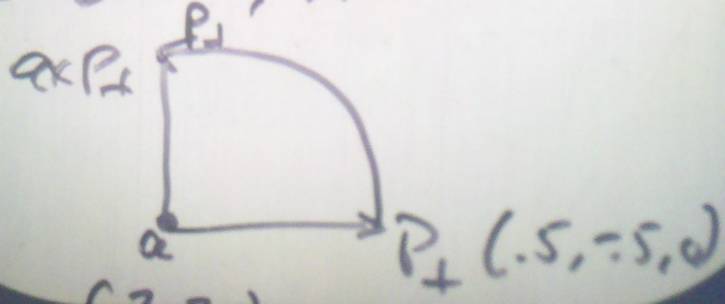
$= (1,0,0) - (.5,.5,0)$

$= (.5,-.5,0)$

ROTATE BY 90°

$P_{\parallel}' = P_{\parallel}$

LOOK DOWN, AXIS a



$(.7, .7, 0)$

$P_\perp' = \cos\theta \, P_\perp + \sin\theta \, a \times P_\perp$

$a \times P_\perp : \begin{pmatrix} i & j & k \\ .7 & .7 & 0 \\ .5 & -.5 & 0 \end{pmatrix}$

$= (0, 0, -.7)$

$P_\perp' = (0, 0, -.7)$

$P' = P_\parallel' + P_\perp'$

$= (.5, .5, 0) + (0, 0, -.7)$

$= (.5, .5, -.7)$

[illegible]

CONVERT:

$a \cdot p\, a$

TO $M_a\, p$ $\qquad m_{i,j} = a_i a_j$

Eg. $a = (.7, .7, 0)$

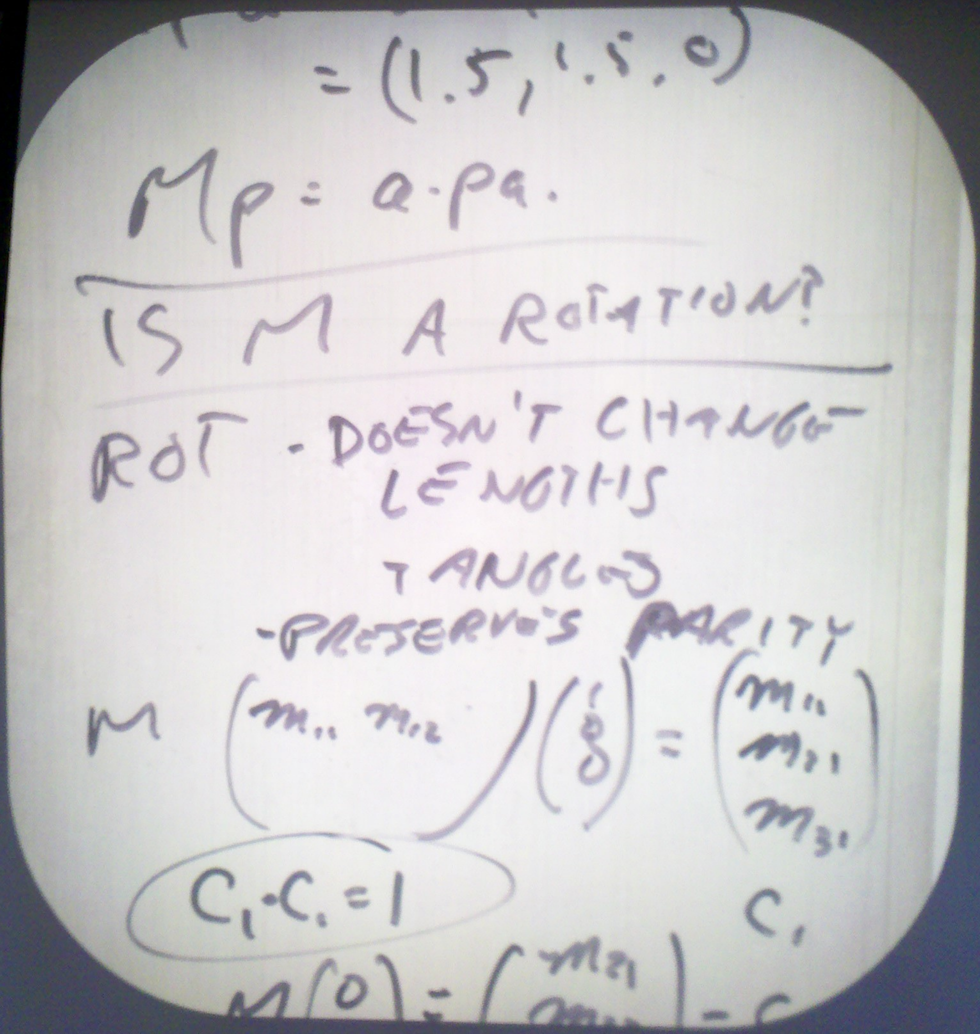
$$M = \begin{pmatrix} .5 & .5 & 0 \\ .5 & .5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1.5 \\ 0 \end{pmatrix}$$

TRY $p = (1, 2, 3)$

$a \cdot p = .7 + 1.4 = 2.1$

$a \cdot p\, a = 2.1\,(.7, .7, 0)$

$= (1.5, 1.5, 0)$



$= (1.5, 1.5, 0)$

$Mp = a \cdot pa.$

IS M A ROTATION?

ROT - DOESN'T CHANGE LENGTHS + ANGLES

- PRESERVES PARITY

$$M \begin{pmatrix} m_{11} & m_{12} & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m_{11} \\ m_{21} \\ m_{31} \end{pmatrix}$$

$C_1 \cdot C_1 = 1$ $\qquad C_1$

$M\begin{pmatrix} 0 \\ \end{pmatrix} = \begin{pmatrix} m_{21} \\ m \end{pmatrix} = C$

$\boxed{C_1 \cdot C_1 = 1}$ $\quad C_1$

$$M\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}m_{21}\\ m_{22}\\ m_{23}\end{pmatrix} = C_2$$

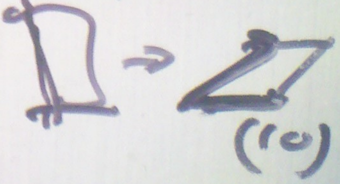
$C_2 \cdot C_2 = 1$

$C_3 \cdot C_3 = 1$

$$\begin{pmatrix} & 1\\ - & 2\\ & 3\end{pmatrix}$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \perp \begin{pmatrix}0\\1\\0\end{pmatrix}$$

2D

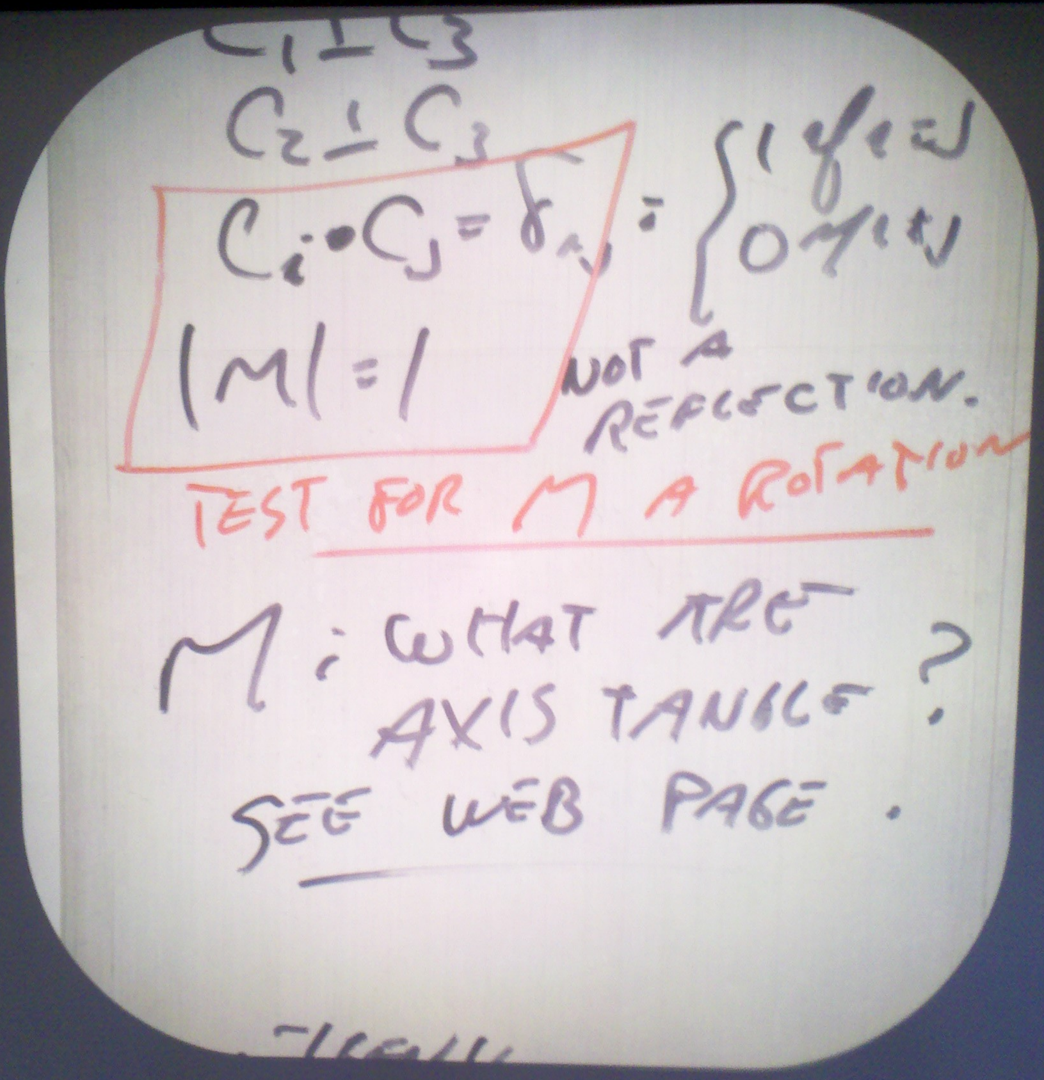


$C_1 \perp C_2$

$C_1 \perp C_3$

$C_2 \perp C_3$

$$\boxed{C_i \cdot C_j = \delta_{ij}} = \begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\end{cases}$$



$C_1 \perp C_3$

$C_2 \perp C_3$

$$\boxed{C_i \cdot C_j = \delta_{ij} \quad |M| = 1} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j \end{cases}$$

NOT A REFLECTION.

TEST FOR M A ROTATION

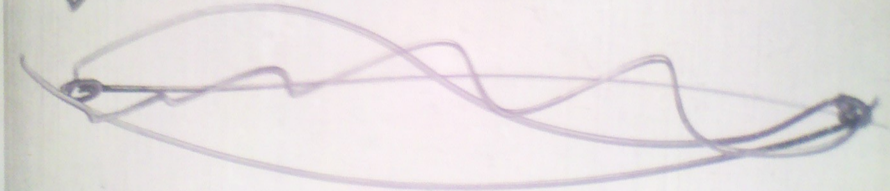
M: WHAT ARE AXIS + ANGLE?

SEE WEB PAGE.

EIGENV.

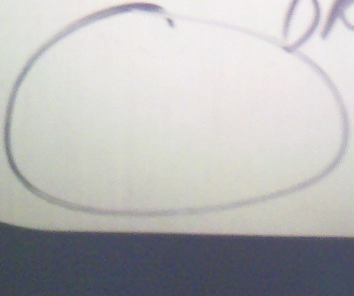
$$M\underline{v} = \lambda \underline{v}.$$

VIOLIN STRINGS

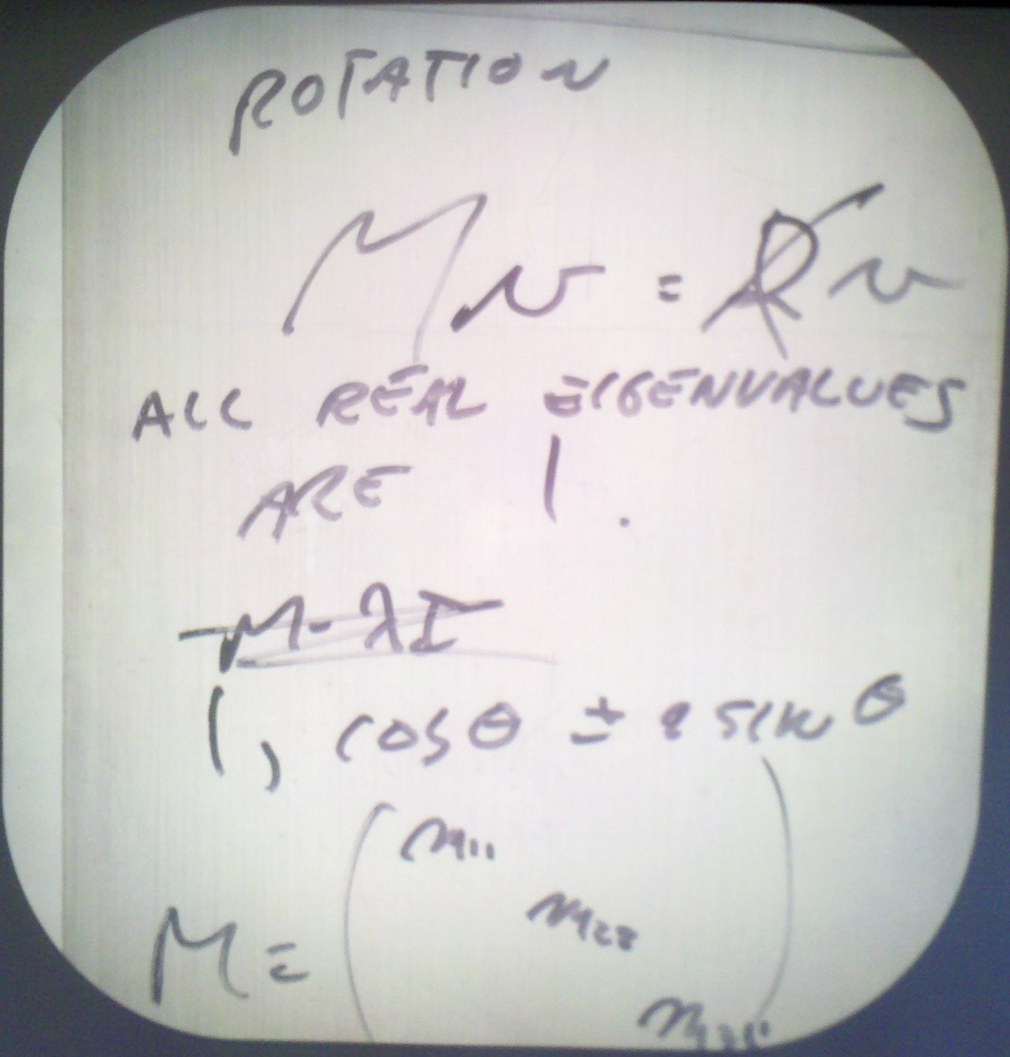


$v, 2v, 3v, 4v.$

DRUM SKIN



$\underline{v}_1 \quad \underline{v}_2 \quad v_3$



ROTATION

$$Mv = \cancel{\lambda} v$$

ALL REAL EIGENVALUES ARE 1.

$\underline{M - \lambda I}$

$1, \cos\theta \pm i \sin\theta$

$$M = \begin{pmatrix} m_{11} & & \\ & m_{22} & \\ & & m_{33} \end{pmatrix}$$

$m_{11} + m_{22} + m_{33}$

$= 1 + 2\cos\theta$

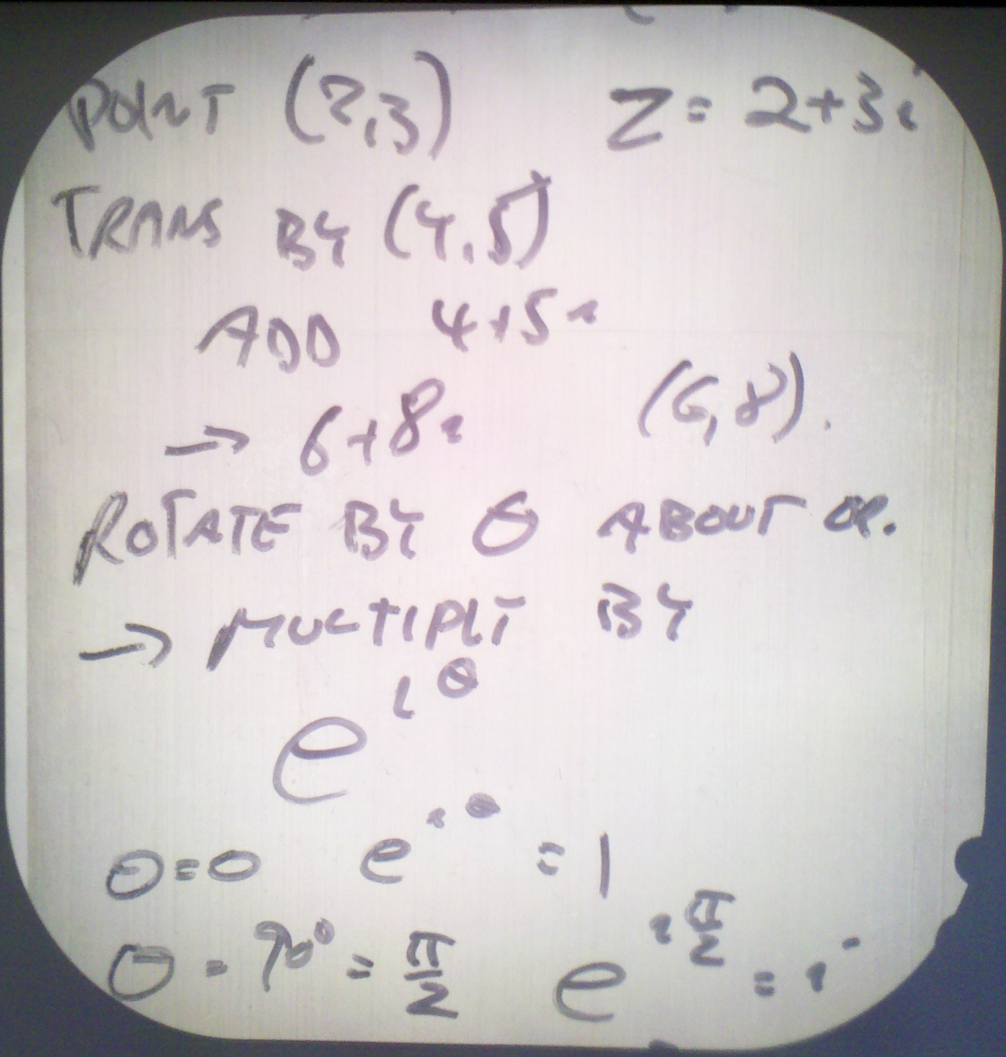
QUATERNIONS

2D POINT (x, y)

COMPLEX NUMBER $z = x + iy$

TRANSLATE BY (x_0, y_0)

$\Longrightarrow$ ADD $(x_0 + iy_0)$



POINT (2,3) $Z = 2+3i$

TRANS BY (4,5)

ADD $4+5i$

$\rightarrow 6+8i$ (6,8).

ROTATE BY θ ABOUT OR.

$\rightarrow$ MULTIPLY BY

$$e^{i\theta}$$

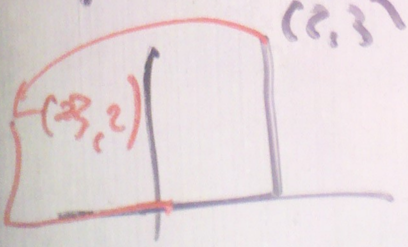
$\theta = 0$ $e^{i0} = 1$

$\theta = 90^\circ = \frac{\pi}{2}$ $e^{i\frac{\pi}{2}} = i$

$\Theta = 180° = \pi \qquad e^{i\pi} = -1$

ROTATE BY 90° → MULT. BY i

$P = (2,3) = 2+3i$



$$(2+3i)i = 2i + 3i^2$$
$$= 2i - 3$$

$(-3,2)$

ROT BY 45° →

MULT BY

$e^{i\pi/4} = (.7 + .7i)$

ROT (2,3) BY 45°

$(2+3i)(.7+.7i)$

$= 1.4 + 2.1i + 1.4i - 2.1$

$= -.7 + 3.5i$

ROT BY 45° →

MULT BY

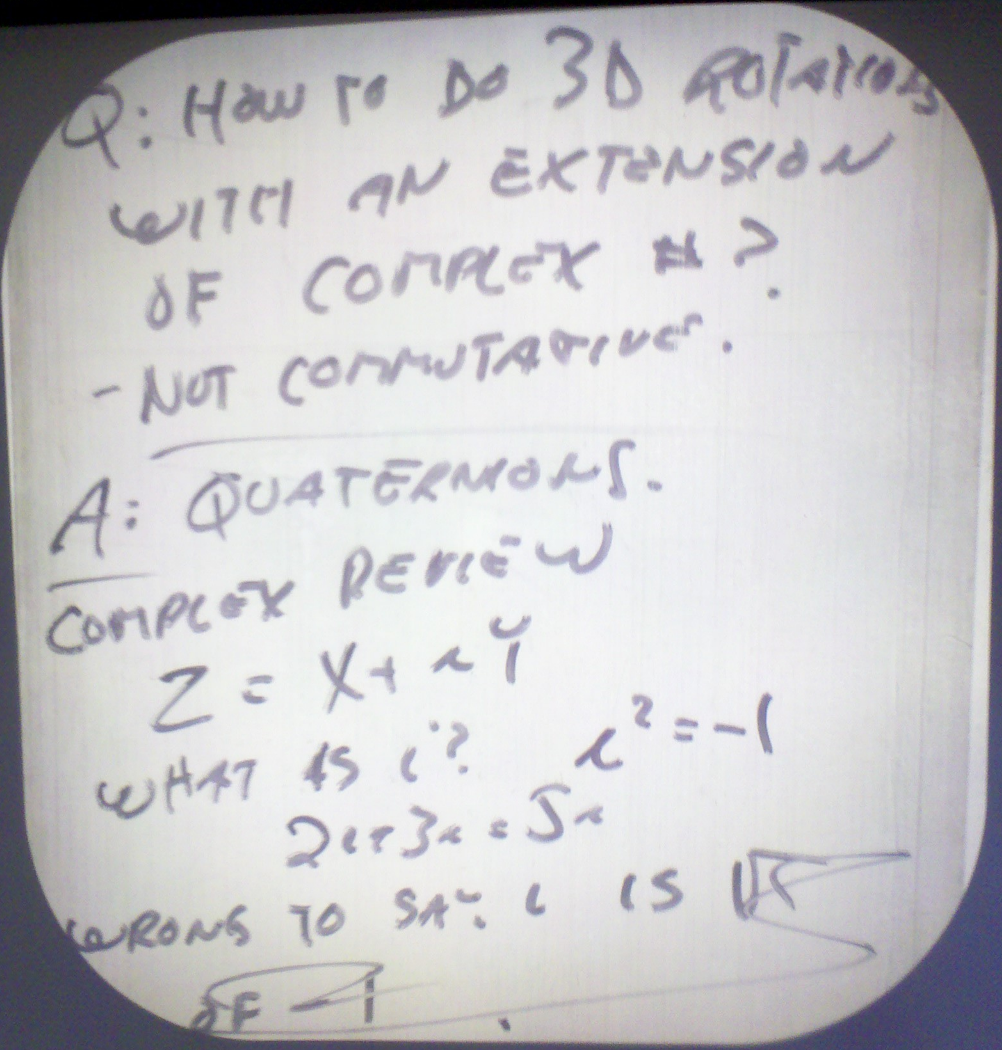
$e^{i\pi/4} = (.7+.7i)$

ROT (2,3) BY 45°

$(2+3i)(.7+.7i)$

$= 1.4+2.1i+1.4i-2.1$

$= -.7+3.5i$



Q: HOW TO DO 3D ROTATIONS
WITH AN EXTENSION
OF COMPLEX #?
- NOT COMMUTATIVE.

A: QUATERNIONS.

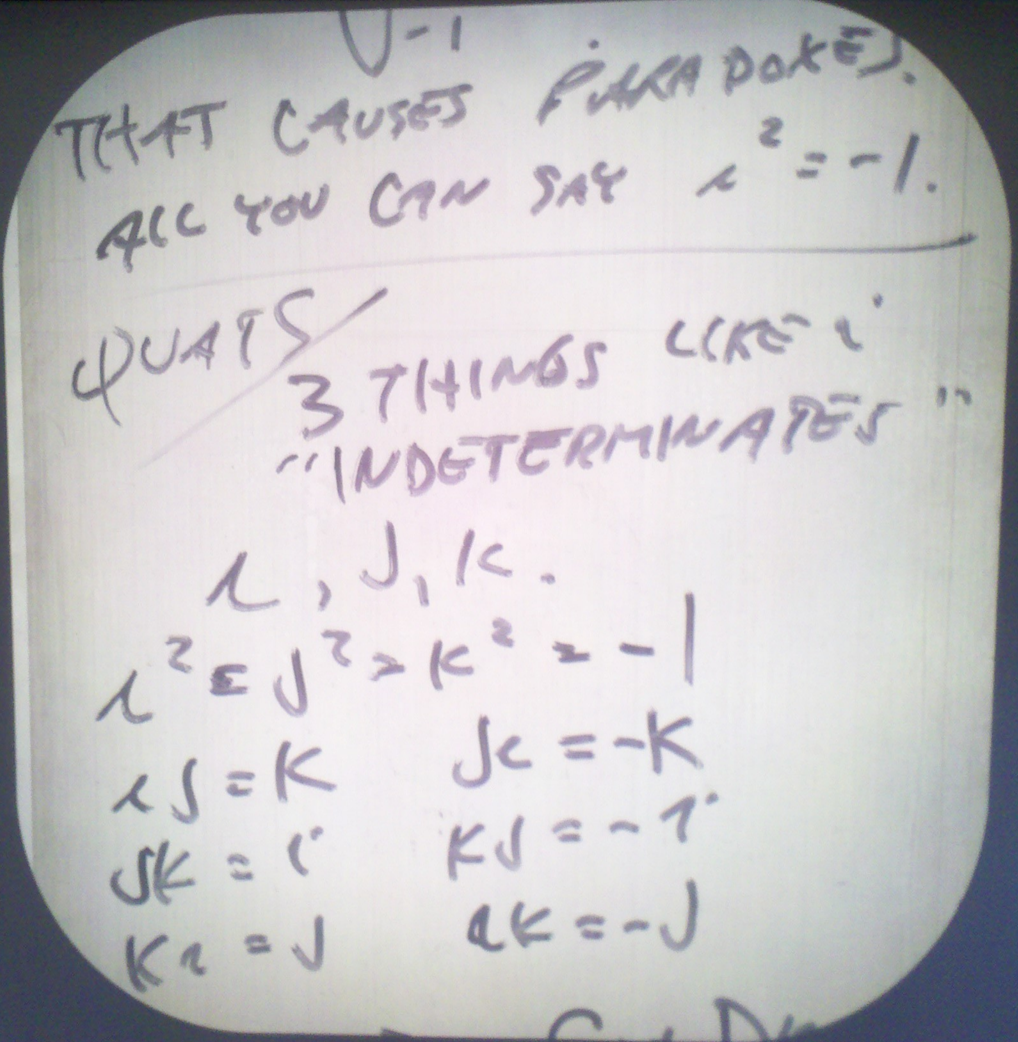
COMPLEX REVIEW

$Z = X + i y$

WHAT IS i? $i^2 = -1$

$2i + 3i = 5i$

WRONG TO SAY: i IS √
OF −1.



$\sqrt{-1}$

THAT CAUSES PARADOXES.

ALL YOU CAN SAY $i^2 = -1$.

QUATS /

3 THINGS LIKE i

"INDETERMINATES"

i, j, k.

$i^2 = j^2 = k^2 = -1$

$ij = k \qquad ji = -k$

$jk = i \qquad kj = -i$

$ki = j \qquad ik = -j$

$$Q = A + Bi + Cj + Dk$$

$$Q_1 = 1 + 2i + 3j + 4k$$

$$Q_2 = i.$$

$$Q_1Q_2 = i + 2i^2 + 3ji + 4ki$$

$$= i - 2 - 3k + 4j$$

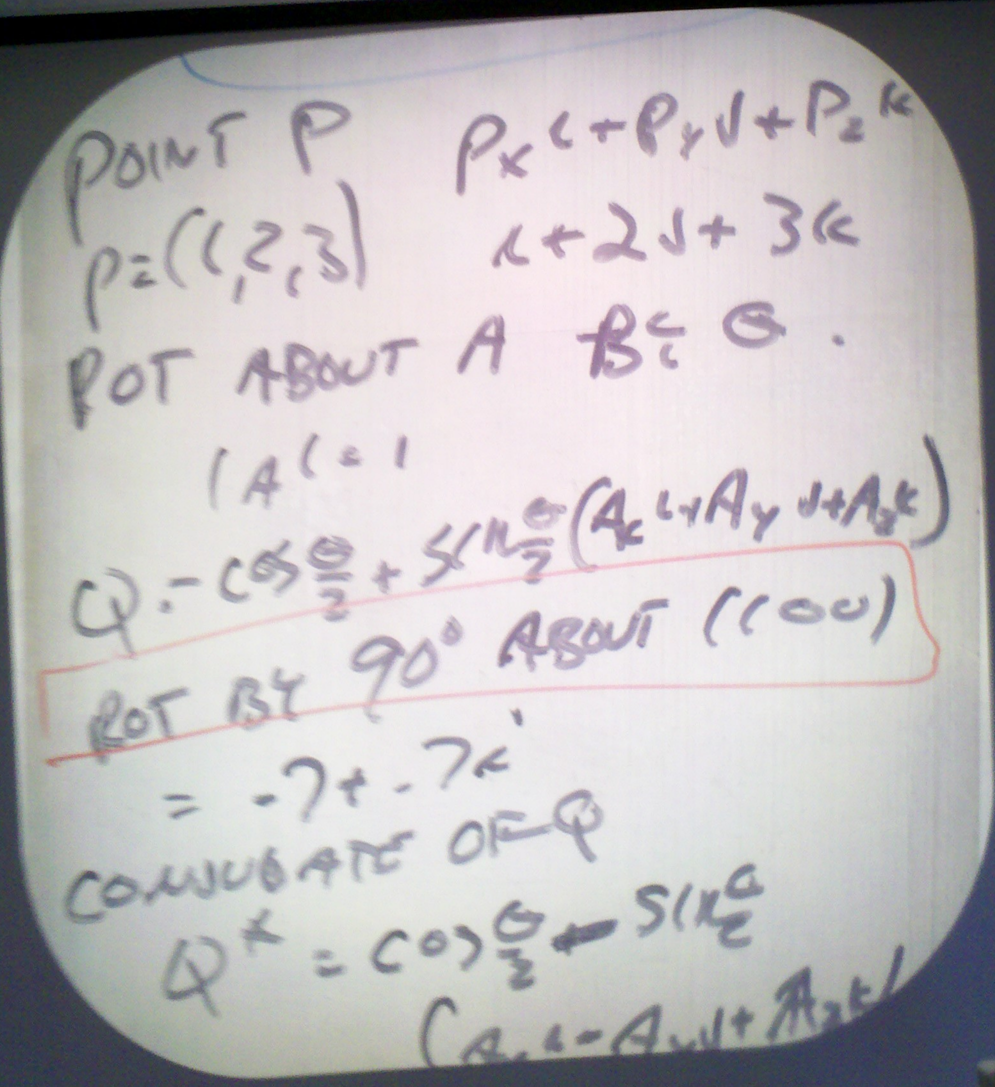
$$\boxed{= -2 + i + 4j - 3k}$$

$$Q_2Q_1 = i(1 + 2i + 3j + 4k)$$

$$= i + 2i^2 + 3ij + 4ik$$

$$= i - 2 + 3k - 4j$$

$$\boxed{= -2 + i - 4j + 3k}$$



POINT P $\quad P_x i + P_y j + P_z k$

$P = (1, 2, 3) \quad i + 2j + 3k$

ROT ABOUT A BY θ.

$|A| = 1$

$Q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(A_x i + A_y j + A_z k)$

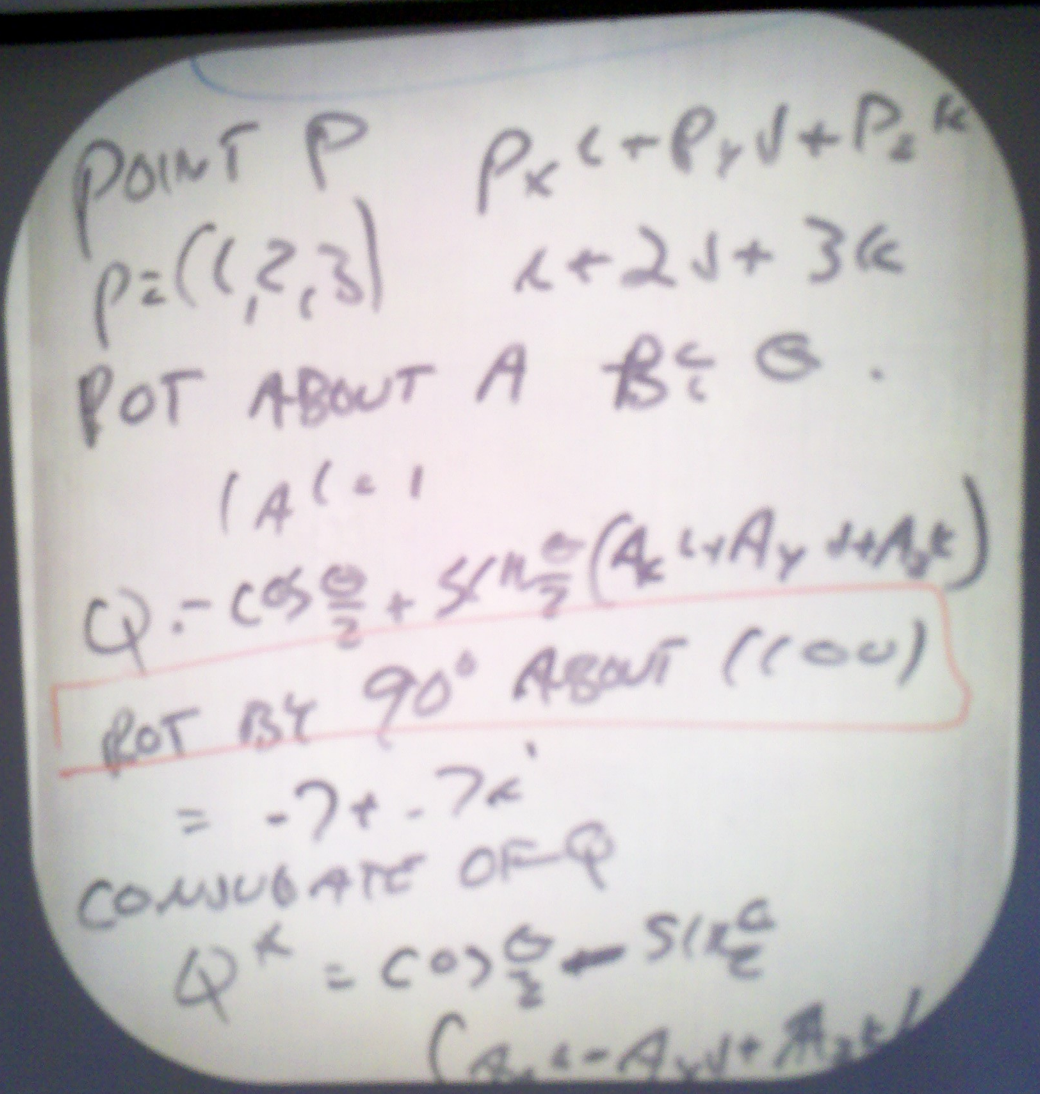
$\boxed{\text{ROT BY } 90^\circ \text{ ABOUT } (1\,0\,0)}$

$= .7 + .7i$

CONJUGATE OF Q

$Q^* = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}$

$(A_x i - A_y j + A_z k$



POINT P $P_x i + P_y j + P_z k$

$P = (1, 2, 3)$ $i + 2j + 3k$

ROT ABOUT A BY θ.

$|A| = 1$

$Q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(A_x i + A_y j + A_z k)$

$\boxed{\text{ROT BY } 90° \text{ ABOUT } (1\ 0\ 0)}$

$= .7 + .7i$

CONJUGATE OF Q

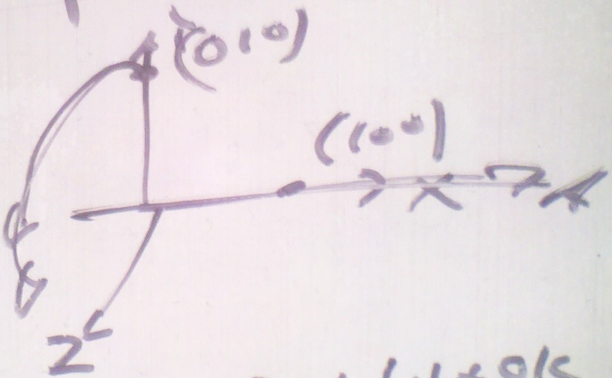
$Q^* = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}$

$(A_x i - A_y j + A_z k$

$(A_x i - A_y j + A_z k)$

$P' = QPQ^*$

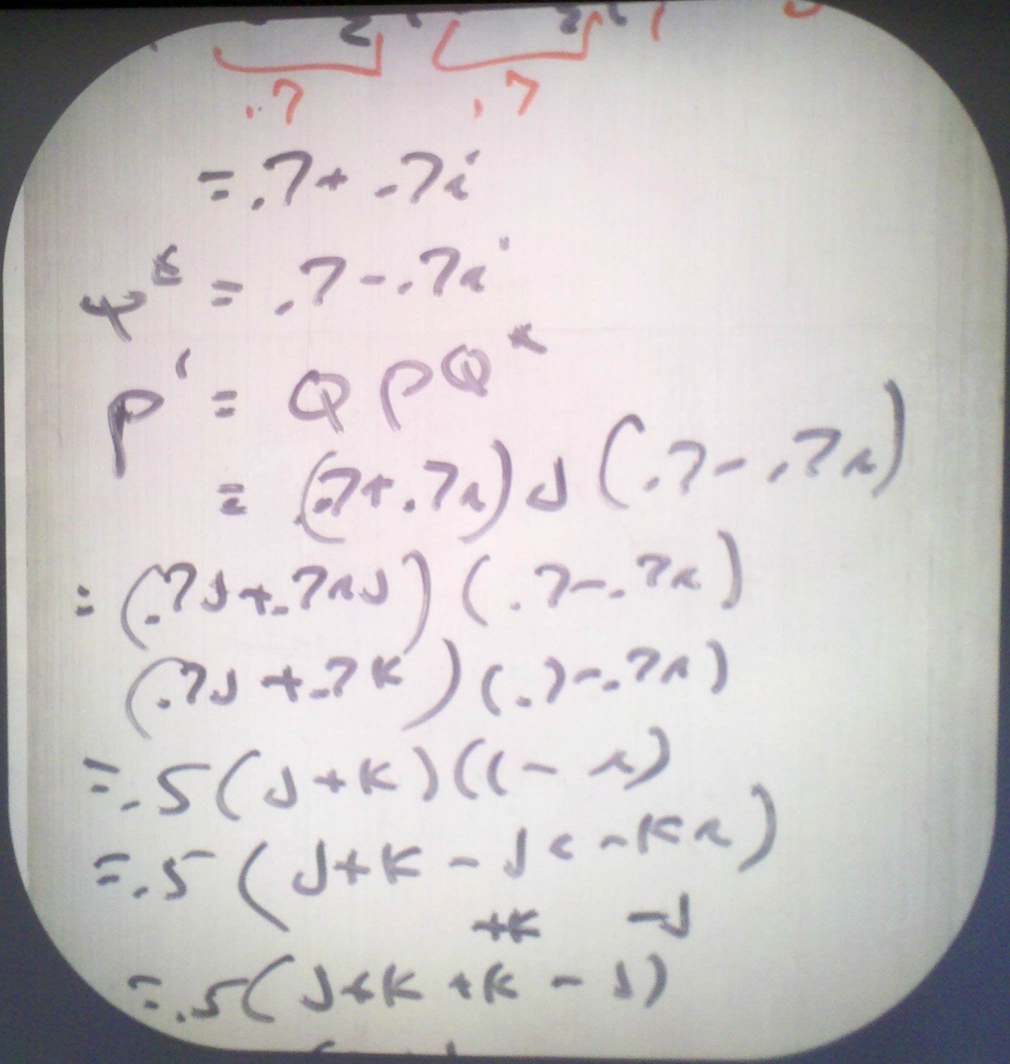
TRY $P = (0,1,0)$



$P = (010) \rightarrow 0i + 1j + 0k$

$= j$

$$Q = \underbrace{\cos\frac{\overset{90}{\theta}}{2}}_{.7} + \underbrace{\sin\frac{\theta}{2}}_{.7}(\underset{1}{a_x} i + \underset{0}{a_y} j + \underset{0}{a_z} k)$$



.7 .7

$= .7 + .7i$

$q^* = .7 - .7i$

$p' = q p q^*$

$= (.7 + .7i) j (.7 - .7i)$

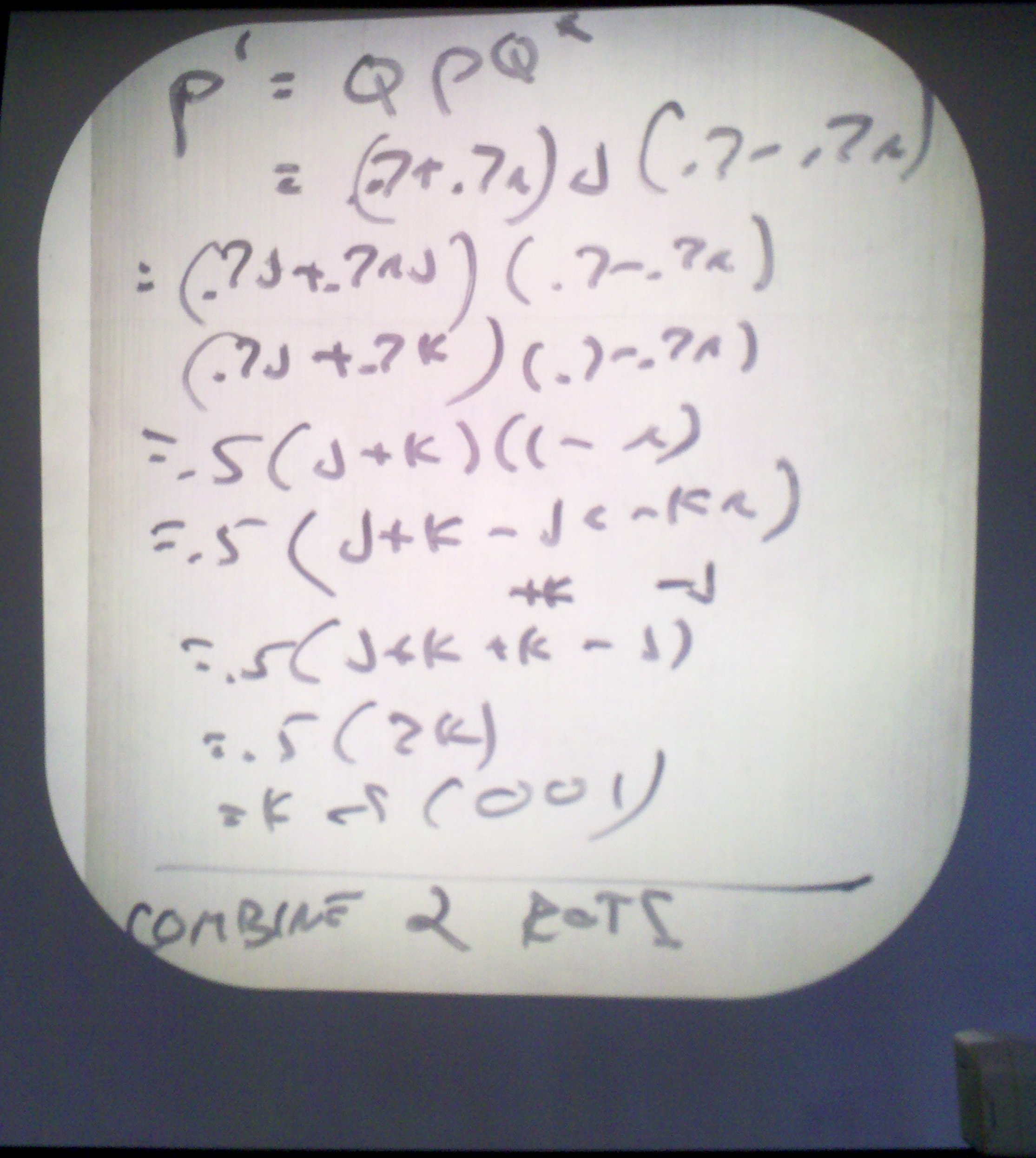
$= (.7j + .7ij)(.7 - .7i)$

$(.7j + .7k)(.7 - .7i)$

$= .5(j + k)(1 - i)$

$= .5(j + k - \underset{+k}{ji} - \underset{-j}{ki})$

$= .5(j + k + k - j)$



$$p' = Q\,p\,Q^*$$

$$= (.7 + .7i)\,j\,(.7 - .7i)$$

$$= (.7j + .7ij)(.7 - .7i)$$

$$(.7j + .7k)(.7 - .7i)$$

$$= .5(j + k)(1 - i)$$

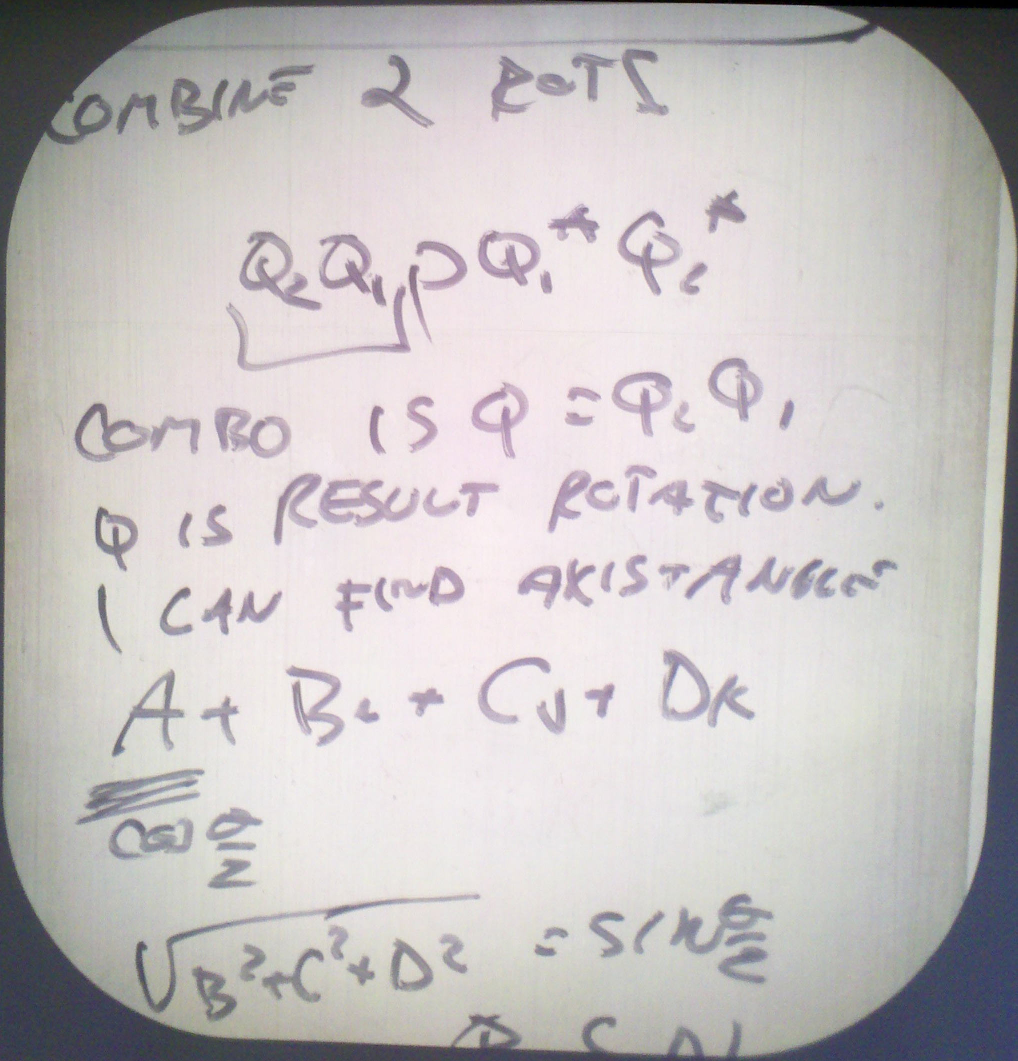
$$= .5(j + k - \underset{+k}{ji} - \underset{-j}{ki})$$

$$= .5(j + k + k - j)$$

$$= .5(2k)$$

$$= k = (0\,0\,1)$$

COMBINE 2 ROTS



COMBINE 2 ROTS

$$\underbrace{Q_2 Q_1}\, P\, Q_1^* Q_2^*$$

COMBO IS $Q = Q_2 Q_1$

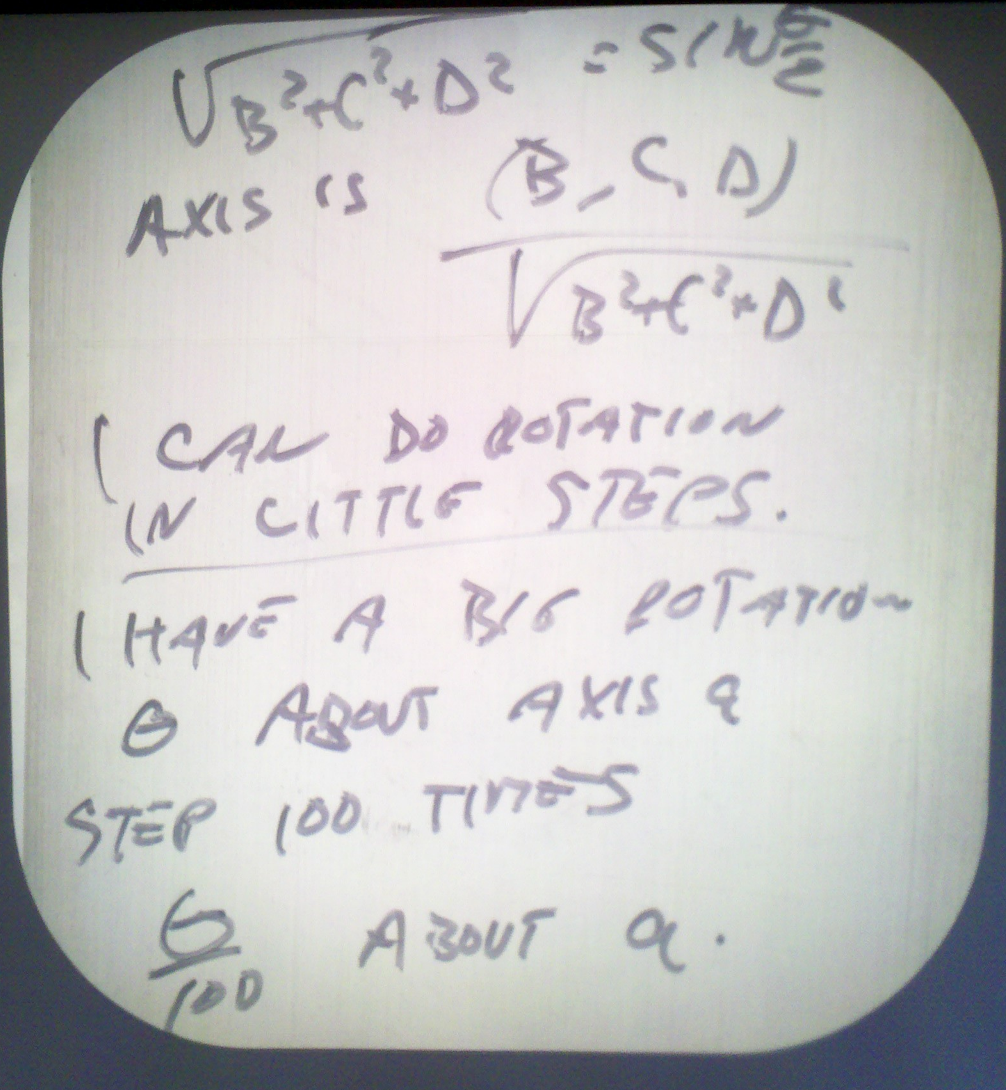
Q IS RESULT ROTATION.

I CAN FIND AXIS+ANGLE

$A + Bi + Cj + Dk$

$\cos\frac{\theta}{2}$

$\sqrt{B^2+C^2+D^2} = \sin\frac{\theta}{2}$



$\sqrt{B^2+C^2+D^2} = \sin\frac{\theta}{2}$

AXIS IS $\frac{(B, C, D)}{\sqrt{B^2+C^2+D^2}}$

(CAN DO ROTATION
IN LITTLE STEPS.

(HAVE A BIG ROTATION
θ ABOUT AXIS a

STEP 100 TIMES

$\frac{\theta}{100}$ ABOUT a.

NOT EASY WITH MATRIX /

WOULD NEED $M^{\frac{1}{100}}$

CAN DO WITH EULER ANGLES
BUT RESULT LOOK BAD.

~~RZ~~ $$\underbrace{R_z\left(\frac{\gamma}{100}\right) R_y\left(\frac{\beta}{100}\right) R_x\left(\frac{\alpha}{100}\right)}_{\text{NOT} = M^{\frac{1}{100}}}$$

HOMOGENEOUS COORDS

$(x\ y\ z\ \underline{1})$