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A Workbench to Compute Unobstructed Shortest Paths in Three-Space

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Abstract

Recently, the following problem has gained considerable importance in computational geometry:

> FINDPATH: Given a set of obstacles and two points (source and goal), calculate the shortest path between these points under the Euclidean metric, constrained to avoid intersections with the given obstacles.

We present three implementations to solve specific instances of FINDPATH in three-space where the obstacles become polyhedra. In the first two cases there exists a single convex polyhedron and the source and the goal points are on its boundary or exterior. These solutions make use of planar developments of polyhedra and polyhedral visibility. The last case is based on a locus method. It partitions the boundary of a convex polyhedron given only the source on it so that for a later goal on the boundary, the shortest path is found efficiently. It makes use of standard point location algorithms for a straight-edge planar subdivision once the partitioning is done.

0. Introduction and Recent Algorithms in Three-Space

Let $P = \{P_1, \cdots, P_n\}$ be a prescribed set of disjoint polyhedra and $s, g \in R^3$ be distinct points which are *not* internal to any P_i. The class of rectifiable curves which have endpoints s and g and which do not intersect any $\mathrm{int}P_i$ will be denoted by $C(s, g; P)$. For C in this class, $\mathrm{l}(C)$ will denote the

length of C under the Euclidean (L_2) metric. An interesting problem in computational geometry asks for the shortest one among these curves:

> FINDPATH
> INSTANCE: Polyhedra $P = \{P_1, \cdots, P_n\}$ such that $P_i \bigcap P_j = \Phi$, $i \neq j$, and $s, g \epsilon R^3$ such that $s \neq g$ and s, g do not belong to $\text{int}P_i$, $1 \leqslant i \leqslant n$.
> QUESTION: Which $C \epsilon C(s, g; P)$ has the shortest length?

The following theorem must be intuitively clear:

THEOREM 0.1 There exists a $C' \epsilon C(s, g; P)$ such that for all $C \epsilon C(s, g; P)$, $l(C') \leqslant l(C)$. Moreover, every such C' is a polygonal path with its possible bend points belonging to some edges of some members of P.
Proof . Omitted. □

It is noted that the polygonal path C' found in theorem 0.1 is not necessarily unique. (In fact, there may be exponentially many shortest paths in terms of n.) We shall call any such path an s-to-g *shortest path*. Thus, FINDPATH asks for the characterization (i.e., the determination of the bend points) of an s-to-g shortest path.

There is a wealth of material in the general area of *motion planning* which comprises other familiar robotics problems such as FINDSPACE, MAKESPACE, etc. in addition to FINDPATH. For brevity, we shall refer the reader to a recent work by Akman[1] which deals with FINDPATH in greater detail and lists numerous references. Franklin and Akman also inspect the same problem from various perspectives in [8, 9, 10, 11].

Sharir and Schorr's work[32] is another detailed study on shortest paths. They mainly consider the case of BOUNDARY FINDPATH (locus) (cf. section 3) and present an algorithm which works in time $O(n)$ per query where n is the measure of complexity of the polyhedron, say the number of vertices. Their algorithm is based on the idea of "ridge" points on the object. A ridge point on the polyhedron has the property that there exists at least two shortest paths to it from the source. It turns out that the set of ridge points is a union of line segments and that the union of the vertices and the ridge points is a closed connected set. Defining the union of the latter with the shortest paths from the source to every vertex, one partitions the polyhedron boundary into disjoint connected regions (called "peels") whose interiors are free of vertices or ridge points. The peels can be constructed in $O(n^3 \log n)$ (preprocessing) time using complicated techniques whose implementation seems extremely difficult. The size of the data structure that is created by the preprocessing step is $O(n^2)$.

O'Rourke et al[24] use Sharir and Schorr's ideas to extend the problem to a polyhedral surface which is no more supposed to be convex. (However, it should be orientable.) The shortest path that they calculate is only a "geodesic", i.e., it is confined to the surface and thus may not be the true shortest path. Their algorithm runs in $O(n^5)$ time. Furthermore, it does not follow a locus approach; using their algorithm on each new query (goal point) would take $O(n^4)$ time.

Mitchell and Papadimitriou[23] give an algorithm to solve this last problem in a locus setting. (It is noted that they are still computing geodesics, not true shortest paths.) Theirs is an $O(n^2 \log n)$ time algorithm for subdividing the surface of an arbitrary polyhedron (possibly of positive genus) so that the length of the shortest path from a given source to any goal on the surface is obtainable by simple point location. It has striking similarities to Dijkstra's method for shortest paths in graphs. As in our algorithm to be presented in section 3, point location is achieved in time $O(\log n)$, after which the actual shortest path is backtracked in time proportional to the number of faces that it traverses on the boundary.

1. Shortest Paths on a Convex Polyhedron

Consider the following specific instance of FINDPATH:

> BOUNDARY FINDPATH
> INSTANCE: Convex polyhedron P, specified points $s, g \in \mathrm{bd}P$ where $s \neq g$.
> QUESTION: Which $C \in \mathrm{bd}P$ has the shortest length?

Before we solve this problem we give an argument as to its importance. Let $P = \{P_1, \cdots, P_n\}$ be a set of *convex* polyhedra. If we are allowed to compute a reasonably short (but *not* the shortest) s-to-g path then we can pursue the following strategy. Let P' be the subset of P such that every member of P' is intersected by sg. (If a polyhedron is intersected by sg only once then it is not included in P'; thus P' consists of polyhedra intersected by sg at precisely two points.) Applying BOUNDARY FINDPATH to each member of P' we obtain an s-to-g polygonal path which comprises two types of curves: line segments through free space, and polygonal paths along the boundaries of the polyhedra between where the path "lands" from free space and "takes off" again (figure 1). Repeated optimization of this path is possible and will frequently yield a better (with fewer bend points) and shorter path (figure 2) although it is not difficult to come up with cases where repeated optimization might cause clashes.

We shall assume that the boundary representation is used to define a

polyhedron P. In this representation scheme each vertex is defined by its x, y, z coordinates and each face is given as a list of pointers to the vertices, ordered in counterclockwise around the boundary of the face with respect to a point above it. It is convenient to think of vertex labels or face labels as positive integers.

DEFINITION The *face graph (Fgraph)* of a convex polyhedron P is an undirected graph $Fgraph = (FV, FE)$ with unit arc weight, $FV = \{i : F_i$ is a face of $P\}$ and $FE = \{(i, j) : F_i$ and F_j are adjacent$\}$. (Two faces are *adjacent* if they share an edge.) □

EXAMPLE Figure 3(a) shows the face graph of a cube. In figure 3(b), the face graph of a parallelepiped is given to note that *Fgraph* only preserves the adjacency information. Figure 3(c) shows that two faces with a common point only are *not* considered adjacent by this definition. □

DEFINITION Let $C \in \text{bd}P$ be an s-to-g polygonal path. The sequence of faces that C enters defines the *face visit sequence* of C which will be denoted by $\text{fvs}C$. □

Thus $\text{fvs}C$ is a walk in *Fgraph* between the nodes corresponding to faces F_s and F_g, the faces of P containing s and g, respectively. An immediate consequence of the above definition is:

LEMMA 1.1 Let $C' \in \text{bd}P$ be an s-to-g shortest path. Then $\text{fvs}C'$ is a simple walk in *Fgraph*.
Proof. If this is not true then C' enters a face at least twice. Recalling the fact that the faces of P are all convex, we can then further shorten C', a contradiction. □

From now on, all face visit sequences will therefore be assumed to be simple.

In addition to *Fgraph*, a useful construct that will be used by BOUNDARY FINDPATH is the planar development of a given face visit sequence. It is well-known that the boundary of a convex polyhedron has the structure of a planar graph. Therefore, the totality of the faces of P, situated in three-space in certain mutual relationship, can be represented in two-space (specifically the xy-plane) by a system of polygons identified with the faces of P. The relationship between a planar development and the planar polygonal polygonal schema will be apparent after the following description of how to obtain the latter.

In the xy-plane associate with each face of P a polygon having the same metrical form. (Two polygons have the same *metrical form* if they can be

made to coincide by translations and rotations.) Define the *glue* relationship between the pairs of edges of these polygons such that two glued edges come from the *same* edge of P. Figure 4 illustrates this for a cube. Each edge in the planar polygonal schema is glued to exactly one edge.

DEFINITION A *planar development* corresponding to a face visit sequence $1, \cdots, k$ is a union of polygons $F_1, \cdots, F_k$ of the planar polygonal schema of P. In the planar development two polygons F_i and F_{i+1}, $1 \leqslant i < k$ are united along the edge that they are glued to each other, and do not overlap. □

DEFINITION The *image* of a point on a polyhedron under a planar development is the point in the plane that it ends up under the development. □

DEFINITION A planar development is *legal* if the line segment connecting the images of the source and the goal is internal to the development. □

EXAMPLE Figure 5 shows several planar developments computed and drawn by SP, our shortest path workbench (cf. appendix). The objects are as follows. Figure 5(a): cube, figure 5(b): icosahedron, figure 5(c) and (d): dodecahedron. It is noted that the last development is not legal. □

It should be apparent that once a planar development is built, it can be moved to any position and orientation in the plane without changing the intrinsic geometry of the paths. We shall now give a procedural definition of a planar development:

DEFINITION To compute the *planar development* of a face visit sequence $1, \cdots, k$, start with F_1. If affF_1 is parallel to the xy-plane then translate P by a suitable amount so that F_1 is now in the xy-plane; otherwise, let the dihedral angle between affF_1 and the xy-plane be D and and rotate P about the line aff$F_1 \bigcap xy$-plane by D to map F_1 to the xy-plane. The remaining faces F_i are inductively handled as follows. Let e be the common edge of F_{i-1} and F_i whose dihedral angle is D. Rotate P by D about aff$e$ to place F_i to the xy-plane while avoiding overlaps with F_{i-1}'s polygon which is already there. □

Now we are ready for:

Algorithm BOUNDARY FINDPATH
1. Let F_s and F_g be the faces of P including s and g, respectively. Assume that $F_s \neq F_g$; otherwise the shortest path is $C^* = sg$.
2. Let FVS be the set of all simple walks in $Fgraph$ of P, between the nodes corresponding to F_s and F_g. Initialize $vs^* = \Phi$ and $l^* = +\infty$.

3. For each member of FVS do the following steps:

3.1 Compute the planar development corresponding to this face visit sequence. Let s' and g' be the images of s and g in the xy-plane under the same development. (They can be computed along with the planar development.)

3.2 If the development in step 3.1 is not legal then continue with step 3. Otherwise, if $d(s',g') < l'$ then replace vs' with the current face visit sequence, let $l' = d(s',g')$, and continue with step 3.

4. At this point l' is the length of the shortest path and vs' is the face visit sequence that should be used to to compute the shortest path itself. To do this, first compute the planar development of vs' (and s' and g') and intersect $s'g'$ with all pairwise common edges of the polygons in the development. The intersection points in the plane are then easily used to compute the bend points of the shortest path C'. We know from the planar development of vs' the distance of an intersection point from a vertex in the plane. All we need is to identify the vertex of P in three dimensions that led to this vertex. Then marking the point which is away from this vertex the same distance over the edge touched by the shortest path we locate the bend point for one intersection. The others are found completely analogously.

End

An efficient way of checking whether a planar development is legal follows. Let $e_1, \cdots, e_t$ be the sequence of edges that are glued in the development. Then the development is legal if $s'g'$ intersects every e_i. Note that this is always easier than testing if $s'g'$ is internal to the planar development's boundary.

To list the simple walks between two nodes of a graph we can use the algorithm of Yen[36] which works in $O(kn^3)$ if there are n nodes in the graph and we require the first k simple walks in increasing walk length. Katoh et al[15] give an improved algorithm for the same task with a time complexity $O(kf(n,m))$ under the assumption that shortest walks form one node to all others can be found in $f(n,m)$ time where m is the number of arcs in the graph. Since $f(n,m)$ is either $O(n^2)$ or $O(m \log n)$ in the worst case, this algorithm is more efficient than Yen's.

Determining the value of cardFVS is difficult. Garey and Johnson[12] state that the following problem is NP-hard:

K-th SHORTEST PATH
INSTANCE: Graph $G=(V,E)$, positive integer lengths l_e for each $e \in E$, specified nodes $s, t \in V$, positive integers b and k.
QUESTION: Are there k or more distinct simple walks from s to

t in G, each having total length b or less?

They also mention that K-th SHORTEST PATH remains NP-hard even if $l_e = 1$ for all $e \epsilon E$, and is solvable in pseudo-polynomial time (polynomial in card$V$, k, and log$b$) and accordingly, in polynomial time for any fixed k (e.g., Yen's algorithm). The difficulty of K-th SHORTEST PATH basically resides in the following counting problem which was proven to be #P-complete by Valiant[34]:

> S-T PATHS (SELF-AVOIDING WALKS)
> INSTANCE: Graph $G = (V, E)$, specified nodes $s, t \epsilon V$.
> QUESTION: What is the number of walks from s to t that visit every node *at most* once?

The problem of counting (s, t)-walks in (s, t)-planar graphs is also #P complete[28]. (A graph is called *source-sink planar*, or (s, t)-*planar* in short, if it has a planar representation with nodes s and t on the boundary.) Provan[29] states that the approximation problem for (s, t)-walks is unsolved, in the sense that the following problem is open:

> "For *any* fixed $\epsilon < 1$, does there exist a polynomial algorithm which for a given (s, t)-planar graph G, will give an approximation N_0 for the number N of (s, t)-walks in G which satisfies $|N - N_0| < \epsilon N$?"

On the other hand, for *any* fixed $\epsilon > 0$, can it be proven that the above problem is NP-hard? Currently, the only known approximations are to count the minimum length walks or to enumerate as many walks as possible (using a large value of k in Yen's algorithm, for example).

It is not hard to find a convex polyhedron which has an exponential number of simple walks in its *Fgraph* (figure 6). If there are l lateral faces of this pyramid-like object (not counting the small triangular faces) then the number of simple walks between the nodes F_s and F_g in the figure is $\Omega(2^{l/2})$. This result also shows that our BOUNDARY FINDPATH algorithm is of exponential time complexity in the number of faces of P. As mentioned before, there exist polynomial algorithms by other researchers for this problem. Unfortunately, theirs do not seem to admit practical implementations. In the light of this, the algorithm presented in this section is applicable for objects of moderate complexity. We can also try to test only a certain section (e.g., first few in increasing walk length) of the face visit sequences between the source and the goal faces for an object with many faces with the hope that the shortest path is generated by a short face development sequence. The shortest path rendered by the legal developments found among the developments that

these sequences give may be taken as the true shortest path although this is certainly vulnerable to an adversary.

2. Shortest Paths around a Convex Polyhedron

Now we inspect the following variant of BOUNDARY FINDPATH:

> EXTERIOR FINDPATH
> INSTANCE: Convex polyhedron P and points s, g where at least one of them is external to P, $s \neq g$.
> QUESTION: Which $C \in C(s, g; P)$ has the shortest length?

Without loss of generality, we shall treat the case where s and g are both outside P. In this case, the following fact is useful:

LEMMA 2.1 Let $H = \mathrm{conv}(\{s, g\} \bigcup \mathrm{vert}P)$. Then an s-to-g shortest path for EXTERIOR FINDPATH is entirely on $\mathrm{bd}H$.
Proof . Omitted. □

Thus, once H is computed using standard three-space convex hull algorithms, we can apply BOUNDARY FINDPATH to the instance made of H, s, and g. (Preparata and Hong[26] give an algorithm to find the three-dimensional convex hull in $O(n \log n)$ time for n points.) However, there is a slight difficulty with this approach. Assuming that we want to know which edges of the original polyhedron the shortest path touches, we must keep extra information with H, i.e., which vertex of H comes from which vertex of P. Below, we shall give a more direct method to obtain H while keeping this information implicitly using a visibility-based approach. (Sutherland et al[33] give an overview of polyhedral visibility.)

DEFINITION For a convex polyhedron P and a point x external to it, the *silhouette edges* of P are members of

$$\{e : e \in F_1 \text{ and } e \in F_2 \text{ where } F_1 \in F_{vis} \text{ and } F_2 \in F_{invis}\}$$

Here, F_{vis} (resp. F_{invis}) is the set of visible (resp. invisible) faces of P from viewpoint x. □

Clearly, $F_{vis} \bigcap F_{invis} = \Phi$ since a face of a convex polyhedron is either completely visible or completely hidden to an observer. Let $E_{sil,s}$ (resp. $E_{sil,g}$) denote the silhouette edges of P from s (resp. g). It is clear that the faces of H will be the union of three *disjoint* sets:

$$\mathrm{bd}H = F_{tri,s} \bigcup F_{tri,g} \bigcup (F_{invis,s} \bigcap F_{invis,g})$$

Here $F_{tri,s}$ (resp. $F_{tri,g}$) is a set of triangular faces each characterized by an edge of $E_{sil,s}$ (resp. $E_{sil,g}$) and s (resp. g). In essence, these are the lateral faces of a pyramid-like object with (generally nonplanar) basis $E_{sil,s}$ (resp. $E_{sil,g}$) and apex s (resp. g). It is noted that the geometric complexity of object H is the same as with P.

We conclude this section with an algorithm to compute the silhouette edges of P from a point x external to it:

Algorithm SILHOUETTE
1. Compute $F_{vis,x}$, the visible faces of P from viewpoint x, by checking line segments xc_i where c_i is the center of mass of face F_i against all F_j, $j \neq i$ for intersection. $F_{vis,x}$ consists of all F_i which do not cause any intersection.
2. Let the totality of the visible edges of P from x be $E_{vis,x} = \{e : e \epsilon F$ where $F \epsilon F_{vis,x}\}$. Sort $E_{vis,x}$ and eliminate *both* of duplicate members. The remaining edges are precisely the edges of $E_{sil,x}$.
End

Figure 7 demonstrates the working of EXTERIOR FINDPATH on a simple object.

3. Partitioning the Boundary of a Convex Polyhedron

Finally, consider the following variant of BOUNDARY FINDPATH:

> BOUNDARY FINDPATH (locus)
> INSTANCE: Same as in BOUNDARY FINDPATH except that g is not given. However, it is guaranteed that, when specified, g will be on $\mathrm{bd}P$.
> QUESTION: Preprocess P so that for any specified $g \epsilon \mathrm{bd}P$ the s-to-g shortest path is computed efficiently.

A practical case which would benefit from BOUNDARY FINDPATH (locus) is as follows. Consider a truck on the surface of a mountain modeled as a convex polyhedron, say, a pyramid. Suppose that the truck is required to carry material from a fixed location on the surface to several points, say, construction sites. Then, it is reasonable to compute the shortest routes for the truck (approximated as a point) more efficiently than can be achieved by repeated applications of BOUNDARY FINDPATH for each specified destination.

This is a powerful paradigm of computational geometry known as the *locus*

approach and is studied in Overmars[25]. In solving a problem using this approach, we are allowed to spend some initial effort (i.e., preprocessing) to construct a data structure which will let us answer future requests (i.e., queries) quickly. To be effective, this assumes two things. First, the number of the query points must be large to validate such an initial effort. Second, the data structure must embody succinctly the locus of the required solution and must enjoy the existence of a fast search procedure to retrieve it.

We shall now summarize one such data structure suitable for solving BOUNDARY FINDPATH (locus). The data structure is known as the *Voronoi diagram* and was first introduced to computational geometry by Shamos[31]. Let $S=\{x_1, \cdots, x_n\}$ be a subset of R^2. For $1 \leqslant i \leqslant n$ let

$$\mathrm{regn} x_i = \{y : \mathrm{d}(x_i, y) \leqslant \mathrm{d}(x_j, y) \text{ for all } j\}$$

be the *Voronoi region* of point x_i. The Voronoi diagram of S, denoted by $\mathrm{vor}S$, partitions the plane into $\mathrm{card}S = n$ regions, one for each member of S. The (open) Voronoi region of point x_i consists of all points of R^2 closer to x_i than any other point of S. For $1 \leqslant i, j \leqslant n$, letting $H_{ij} = \{y : \mathrm{d}(x_i, y) \leqslant \mathrm{d}(x_j, y)\}$ (the halfspace defined by the perpendicular bisector of $x_i x_j$), it is seen that for $i \neq j$, $\mathrm{regn} x_i = \bigcap H_{ij}$. Thus, $\mathrm{regn} x_i$ is a convex polygonal region and $\mathrm{vor}S$ is equal to the union of the boundaries of all $\mathrm{regn} x_i$. For every vertex x of $\mathrm{vor}S$ there are at least three points x_i, x_j, x_k in S such that $\mathrm{d}(x, x_i) = \mathrm{d}(x, x_j) = \mathrm{d}(x, x_k)$. The Voronoi diagram for a set of n points has at most $2(n-2)$ vertices and $3(n-2)$ edges[22].

The Voronoi diagram of S can be computed in $O(n \log n)$ time [30] and this is optimal with respect to a wide range of computational models[17]. Unfortunately, practical Voronoi programs which are both fast and reliable are difficult to write mainly due to the special cases in the diagram that are to be handled precisely. Among the published algorithms, those by Avis and Bhattacharya[2], Lee[19], and Guibas and Stolfi[14] seem to be more promising for practical use. On the other hand, it is possible to construct slower implementations which handle special cases without much effort.

We now summarize how to search Voronoi diagrams in logarithmic time in the number of the edges, n, of the diagram, i.e., we cite methods which let one find the point $x \in S$ such that for a query point $y \in R^2$, $\mathrm{d}(x, y)$ is minimum. The search methods we shall review are more general than searching Voronoi diagrams in that they are based on searching a *planar subdivision*, i.e., a straight-edge embedding of a planar graph. The underlying problem is generally known as *planar point location* in computational geometry. Let us call a subset of the plane *monotone* if its intersection with any line

parallel to the y-axis is a single interval (possibly empty). A subdivision is monotone if all its regions are monotone. Mehlorn[22] shows that a *simple* planar subdivision (a subdivision with only triangular faces) can be searched in time $O(\log n)$ after spending preprocessing time $O(n)$ and storage space $O(n)$. He also gives the following property to show that the last two bounds apply to *general* subdivisions also, with a small penalty in preprocessing time:

LEMMA 3.1 If the searching problem for simple planar subdivisions with n edges can be solved with search time $O(\log n)$, preprocessing $O(n)$, and space $O(n)$, then the searching problem for general subdivisions with n edges can be solved with the same search time and space but preprocessing $O(n \log n)$. If all faces of the generalized subdivision are convex then $O(n)$ preprocessing is sufficient.
Proof . Mehlorn[22]. (Lee and Preparata[18] also show that an arbitrary subdivision with n edges can be refined to a monotone subdivision having at most $2n$ edges in $O(n \log n)$ time.) □

Now we shall review some planar point location algorithms in the literature which either achieve these bounds or come close. Dobkin and Lipton[3] were the pioneers to obtain an $O(\log n)$ query time but they use $O(n^2)$ space. Preparata[27] modifies their method to prove that $O(n \log n)$ space is sufficient. His solution is implementable. In an important paper Lee and Preparata[18] give an algorithm which is based on the construction of separating chains. Their algorithm achieves $O(n \log n)$ preprocessing and $O(n)$ space yet has a query time of $O(\log^2 n)$. The constants hidden in these expressions are small and make their algorithm practically useful. (Their algorithm works for monotone subdivisions only.) Edelsbrunner and Maurer[4] give a space-optimal solution which works for general subdivisions (and even families of nonoverlapping subdivisions). Their query time is $O(\log^3 n)$. Lipton and Tarjan[20, 21] give a method with $O(\log n)$ query time and $O(n)$ preprocessing and space (and thus optimal in all respects). Although based on a graph separator theorem which has many far-reaching consequences, they admit that their method is of only theoretical interest because of the implementation difficulties. Kirkpatrick[16] gives another method with the same bounds. His method builds a hierarchy of subdivisions and seems to be implementable. Finally, Edelsbrunner et al[5] give a substantial improvement of the technique of Lee and Preparata and again attain the optimal bounds in all three respects. The importance of their method is that it seems to admit an efficient implementation along with extensibility to subdivisions with curved edges.

Now we are ready to present our algorithm for BOUNDARY FINDPATH (locus). In the following we show the partitioning for only a face of P (other than F_s) which we shall denote by F_g. To partition bd$P$, we apply the

following algorithm for each face. (Note that no partitioning is necessary for F_s due to convexity.)

Algorithm BOUNDARY FINDPATH (locus): Preprocessing
1. Find all simple face visit sequences between F_s and F_g using *Fgraph* of P.
2. For each face visit sequence found in step 1, compute the image of s with respect to the planar development which starts with face F_g and ends with F_s. (Note that we are *not* required to compute the whole development, but just the image point.)
3. Compute the Voronoi diagram of the image points calculated in step 2 using standard Voronoi programs mentioned above. It is required that with each image point (Voronoi center) we store the face visit sequence which is used to arrive it.
4. Clip the diagram obtained in step 3 with respect to "window" F_g so as to preserve only those parts of it within the polygon F_g. (Any of the standard graphics algorithms such as the one due to Sutherland and Hodge[7] can be used for clipping.)
End

Assume that for a given P, the above preprocessing is carried out for all faces (except F_s). Thus, we have a family of planar subdivisions for each face such that for each region of a given subdivision we know the ordered sequence of faces to be developed to the plane if g specified within that region. More specifically, we can apply the following algorithm when we are queried with a new g:

Algorithm BOUNDARY FINDPATH (locus): Querying
1. Compute the face F_g holding a given g. If $F_g = F_s$ then the shortest path is trivially sg.
2. Using standard planar point location algorithms mentioned above locate g inside the planar subdivision belonging to F_g.
3. Since we stored the face development sequence used to arrive this region we can now use it to compute the s-to-g shortest path. Note that there may be cases where g will be shared by at least two regions and thus there will be at least two shortest paths each of which is obtained via a different development sequence.
End

Figure 8 shows the Voronoi partitioning on a face of a cube using the above algorithm. This figure was drawn by SP. It is noted that in figure 8(a) the given face is partitioned into 4 regions whereas in figure 8(b) this number becomes 6. This effect was obtained by simply moving the source to another location on the source face.

In general, the number of regions a given face F_g is partitioned by this algorithm is expected to be small. An informal argument for this is as follows. Consider the images of s in the plane of F_g. If the face visit sequence for a particular image is long then the image will usually end up in a point farther away from F_g compared to another image obtained by a shorter visit sequence. Thus, only a small portion of the possible face visit sequences between F_s and F_g can render images in the plane close to F_g and contribute to the Voronoi diagram on it.

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Appendix: SP - A Workbench to Compute Shortest Paths

SP is a family of programs written in Franz Lisp[6] and Macsyma command language[13] to experiment with shortest paths. A detailed description of SP is given in [1].

SP was designed with the following philosophy. Let W be a workspace (e.g., a bounding box) which includes a set of polyhedral obstacles. SP is given a geometric description of the members of W and from that point on should be able to compute shortest paths inside W between given pairs of points using the algorithms some of which presented in the preceding sections. It is imperative that SP has some graphics facilities and can supply the user with views of W so that she can have an intuitive feeling about the correctness of a particular computation. In that sense, SP resembles to Verrilli's system[35]; it provides the user with facilities to carry out needed computations, but at the same time needs her intervention here and there. Following a rapid prototyping approach, we either simply excluded those computations which we do not currently know how to perform effectively, or reformulated them to be controlled by user advice at certain points.

Currently, one can only work with a single convex polyhedron using the Franz part of SP. There are facilities to implement BOUNDARY FINDPATH, EXTERIOR FINDPATH, and BOUNDARY FINDPATH (locus). It is also possible to implement an approximate FINDPATH algorithm for a workspace with several convex polyhedra as outlined in section 1 and depicted in figures 1 and 2. Using Macsyma parts of SP it is possible to compute shortest paths in a general workspace with many objects (which may be nonconvex) although this is not fully automated in the light of the combinatorial

explosion that known FINDPATH algorithms have. (Nevertheless, if the user specifies the list of edges that the shortest path must visit, then the problem is solved without much effort.) It is also possible (using Macsyma) to work on FINDPATH (locus) (which is the general version of BOUNDARY FINDPATH (locus)) although this is not automated yet. (In [11] all computations were done in this way.)

Finally, we give some additional examples computed and drawn by SP. Figure 9(a) and (b) show two shortest paths on the boundary of a dodecahedron. Similarly, figure 10 shows a shortest path on the boundary of an icosahedron. Figure 11 demonstrates a shortest path around a cube. This was computed after constructing a new object via EXTERIOR FINDPATH and then applying BOUNDARY FINDPATH on it. Figure 12 shows the partitioning of the boundary of a cube in the presence of a source point on face 1. Figure 12(a), (b), (c), (d), and (e) respectively depict the regions induced on goal faces 2, 3, 4, 5, and 6.

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Figures

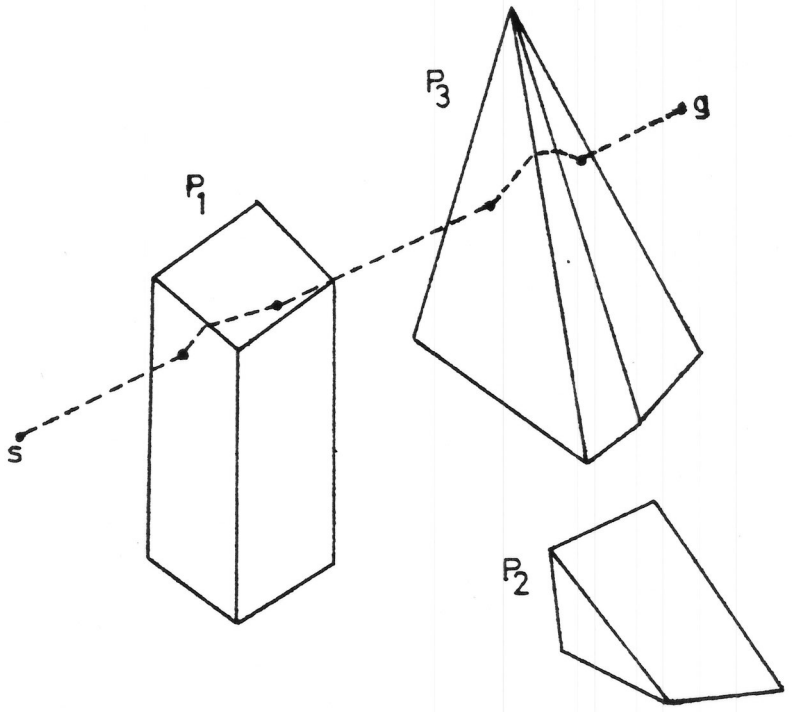


Figure 1 A reasonably short path between s and g in the presence of convex polyhedral obstacles.

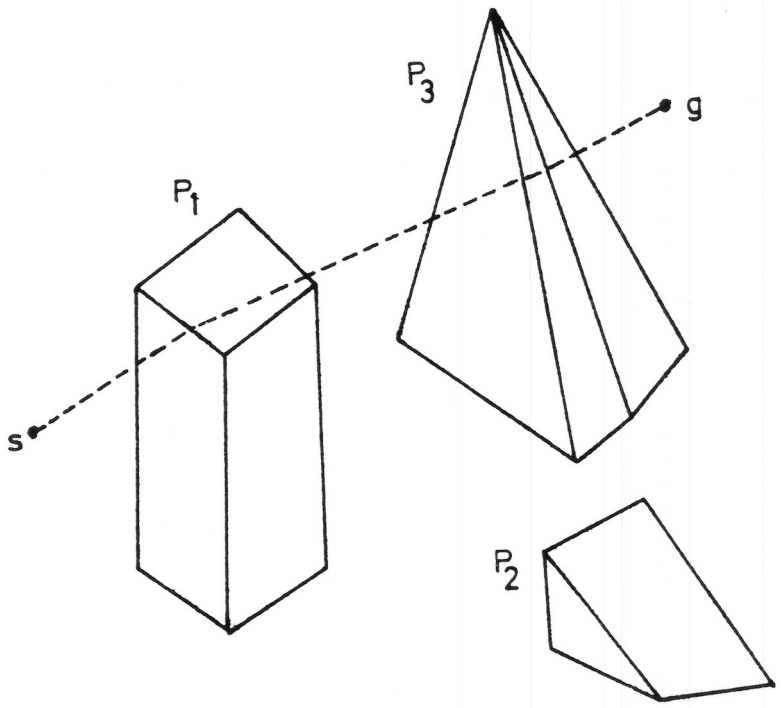


Figure 2 Further optimization of the path shown in figure 1 yields a shorter path with fewer bend points.

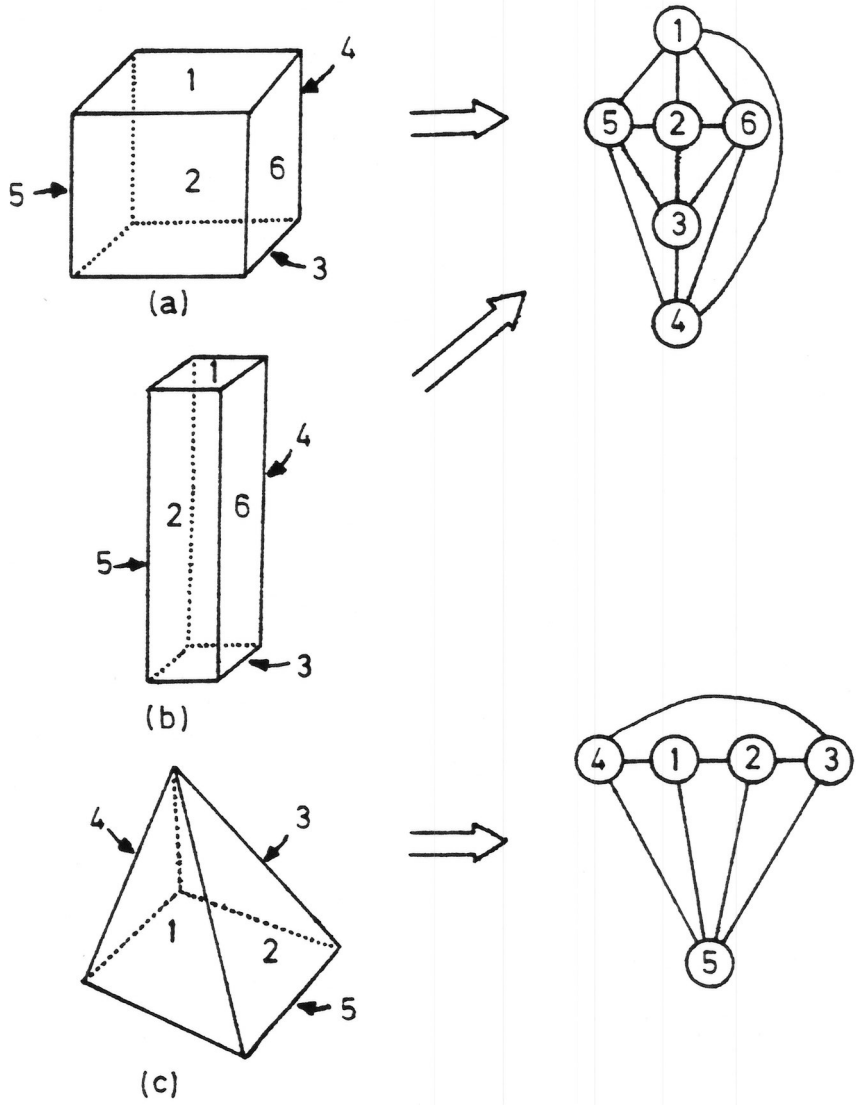


Figure 3 Face graphs of convex polyhedra:
(a) cube, (b) prism, (c) pyramid.

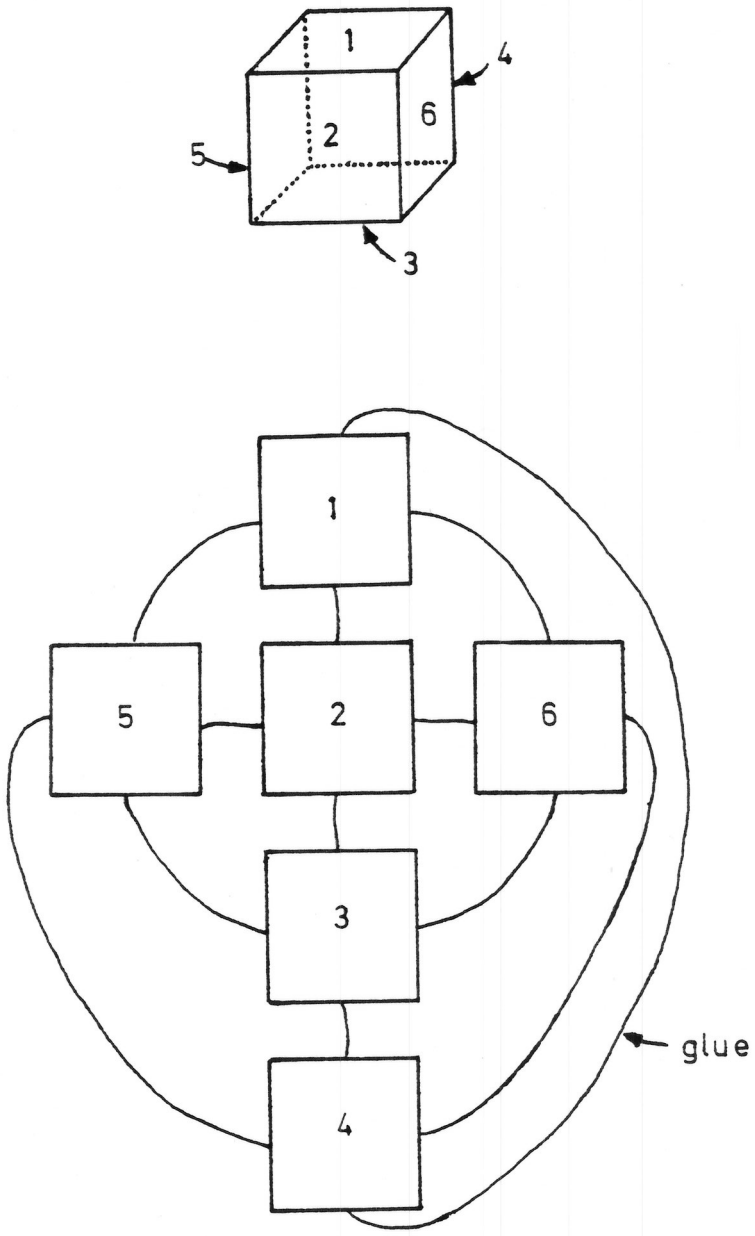


Figure 4 The planar polygonal schema of a cube.

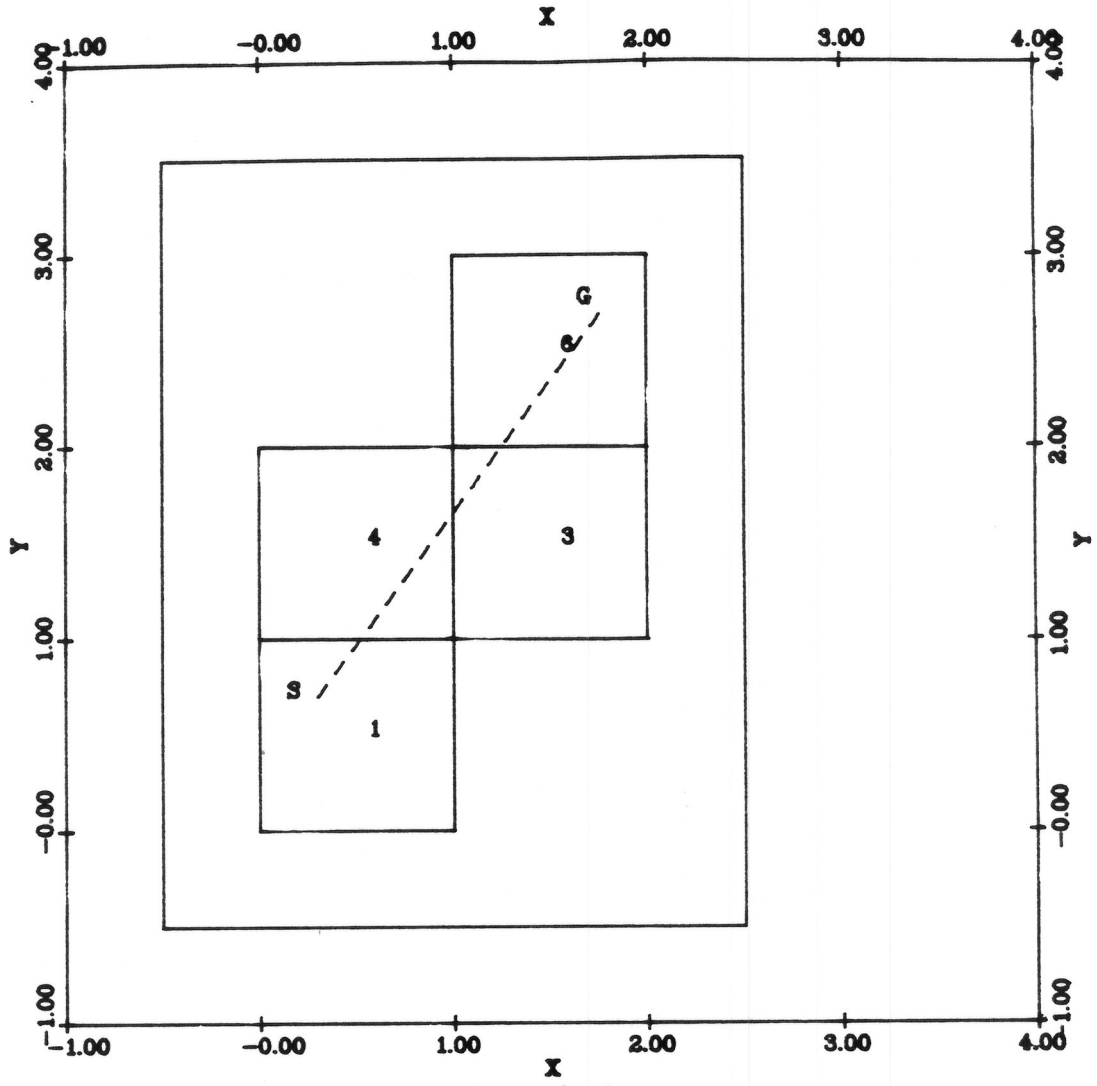
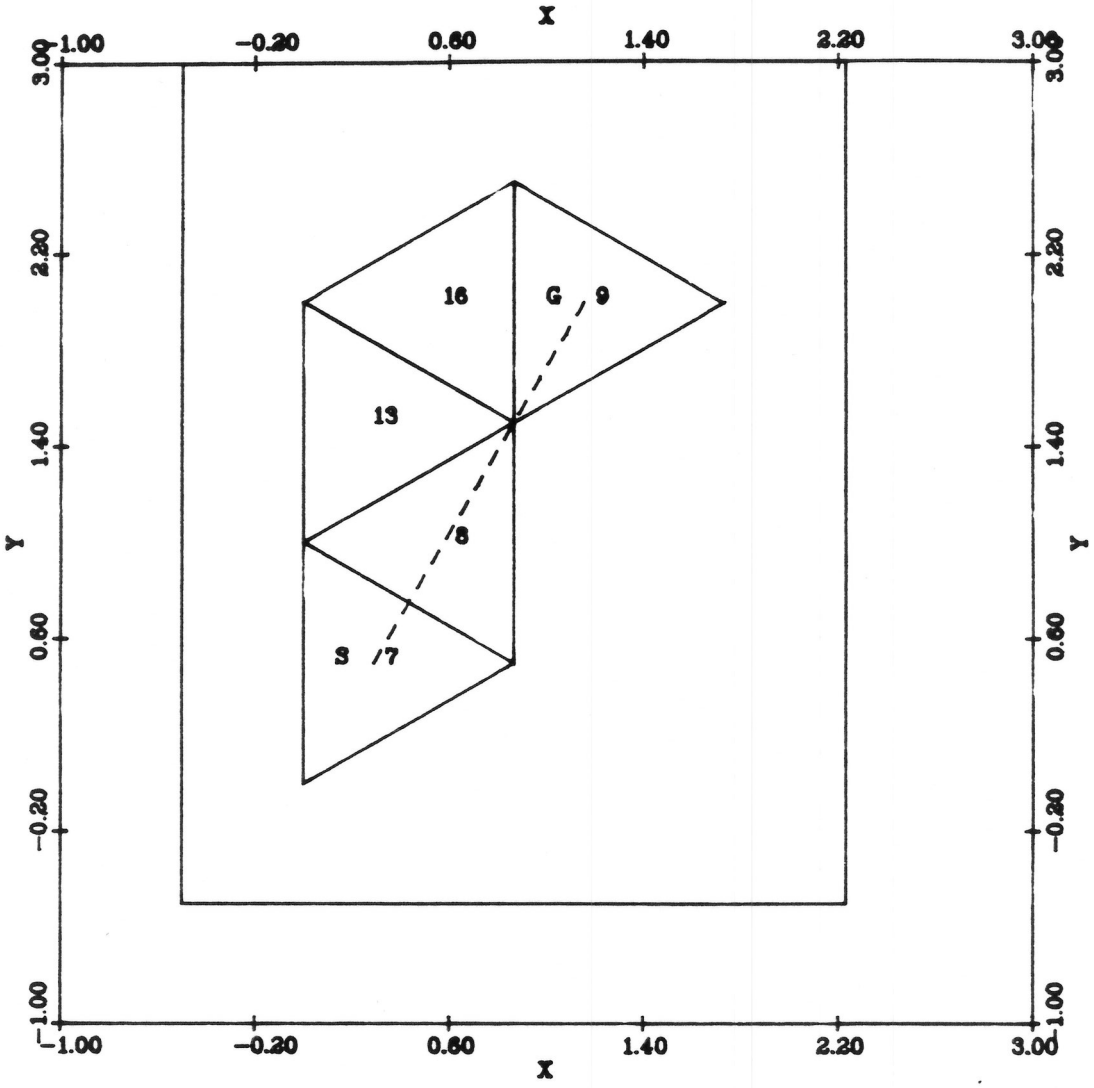
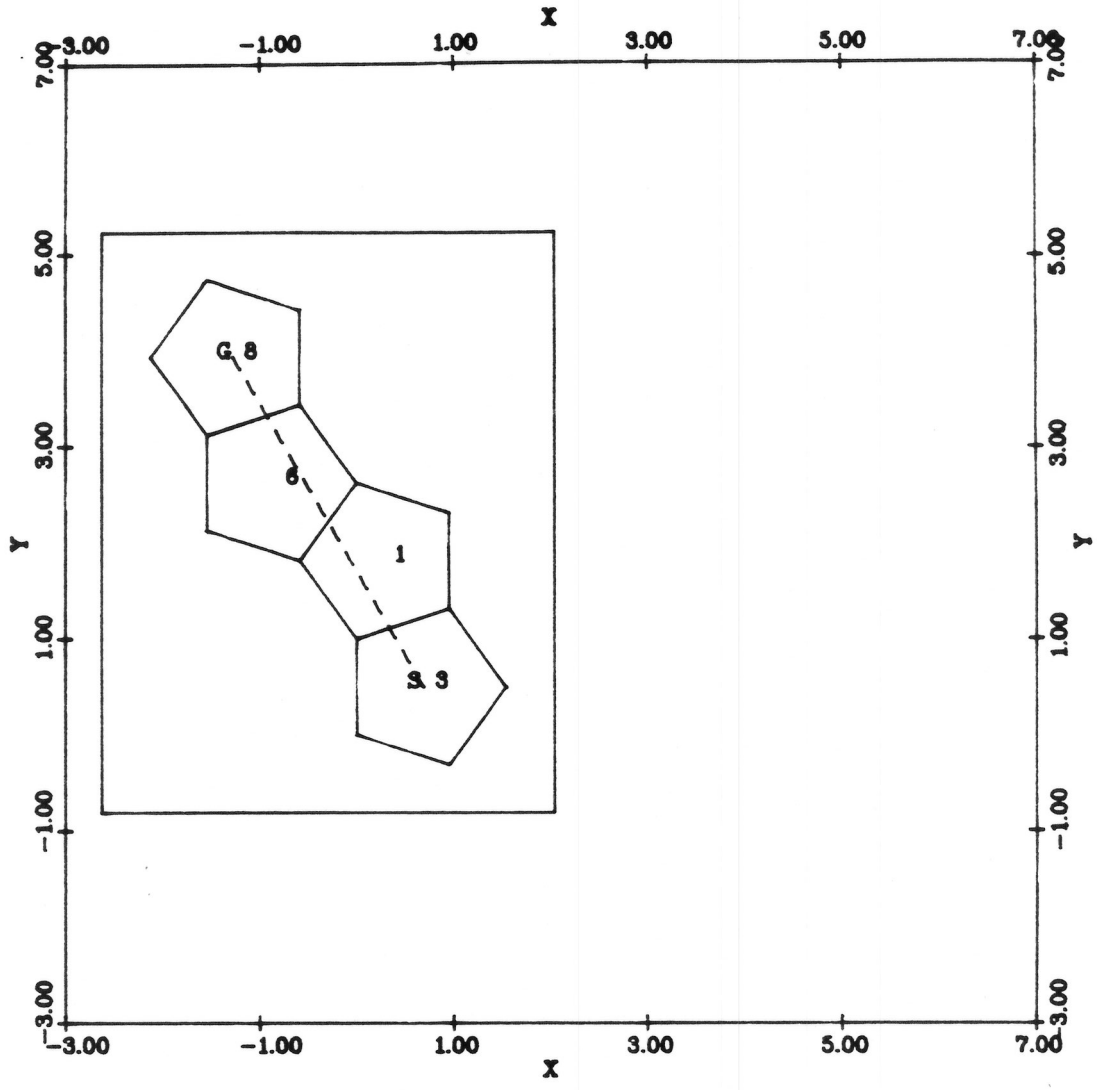


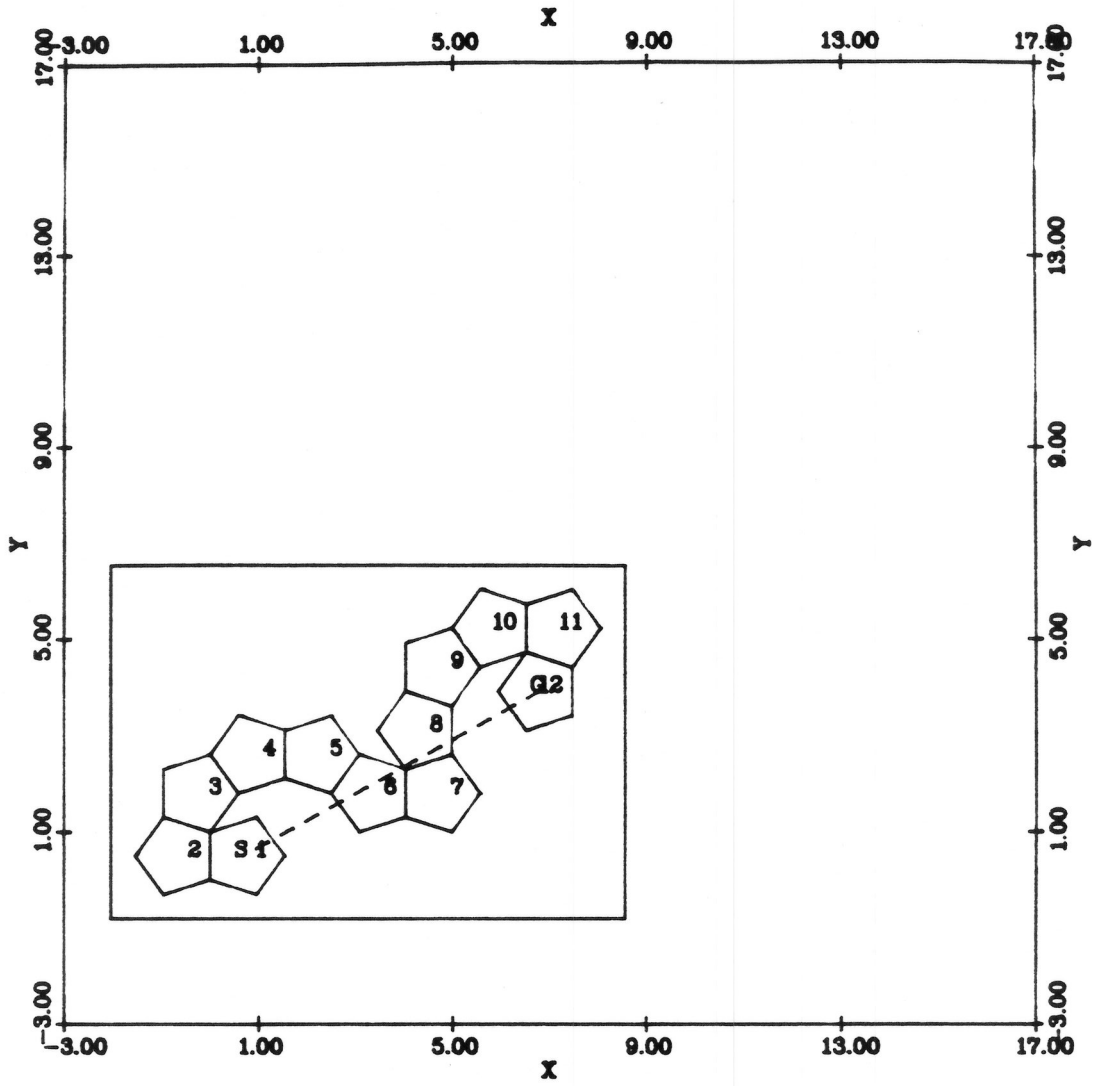
Figure 5 Some planar developments computed and drawn by SP. The objects that are developed are as follows: (a) cube, (b) icosahedron, (c) and (d) dodecahedron.



Face development sequence:	7 8 13 16 9		
Source face no.:	7		
Goal face no.:	9		
Source point coords.:	0.948	0.230	0.504
Goal point coords.:	0.192	1.206	0.504
Source point coords. (plane):	0.289	0.500	
Goal point coords. (plane):	1.154	2.000	
Source-to-goal distance:	1.732		



Face development sequence:	3 1 6 8		
Source face no.:	3		
Goal face no.:	8		
Source point coords.:	0.380	1.447	0.616
Goal point coords.:	0.996	−0.447	1.612
Source point coords. (plane):	0.688	0.500	
Goal point coords. (plane):	−1.276	3.927	
Source-to-goal distance:	3.950		



Face development sequence:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Source face no.:			1									
Goal face no.:			12									
Source point coords.:			0.688		0.500		0.000					
Goal point coords.:			0.688		0.500		2.227					
Source point coords. (plane):			0.688		0.500							
Goal point coords. (plane):			6.782		3.927							
Source-to-goal distance:			6.991									

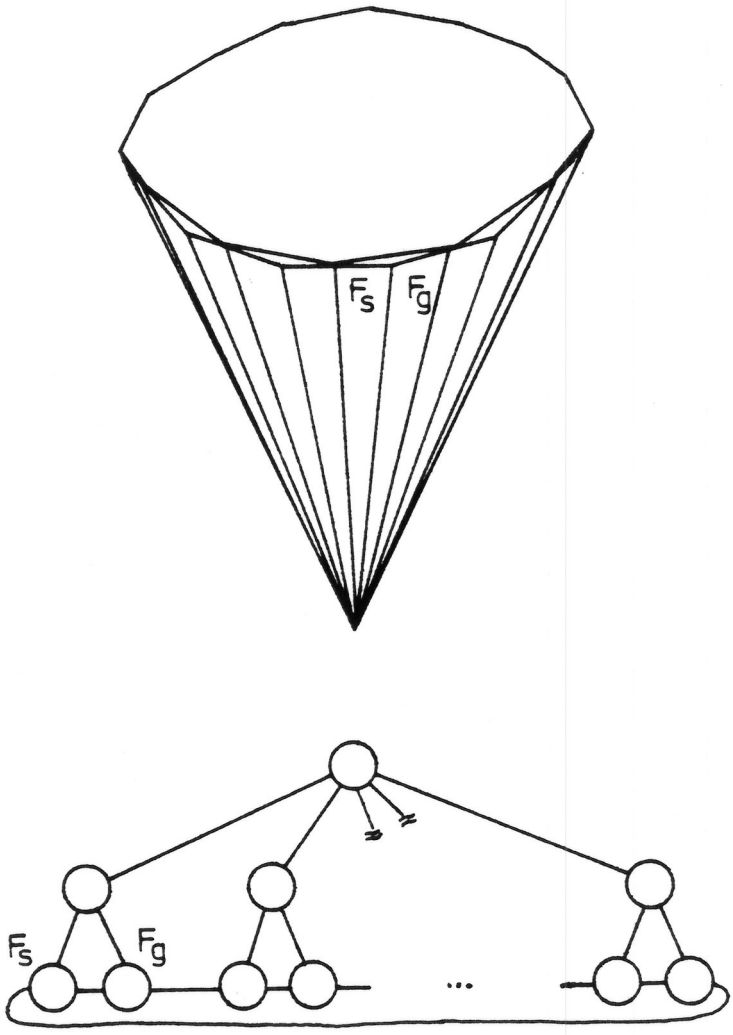


Figure 6 There exist an exponential number of simple walks between nodes F_s and F_g in *Fgraph* of this object.

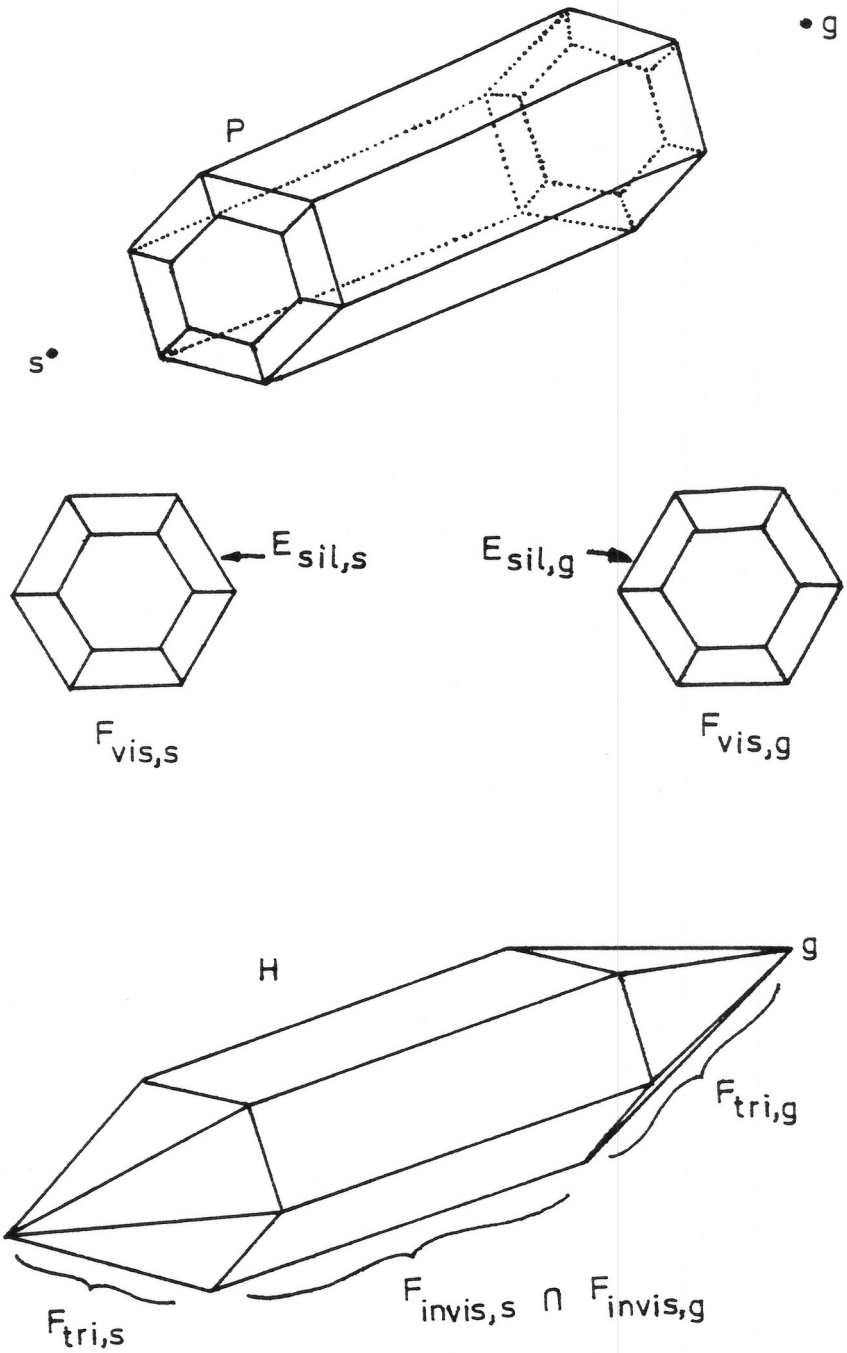


Figure 7 Demonstration of how EXTERIOR FINDPATH works via silhouettes.

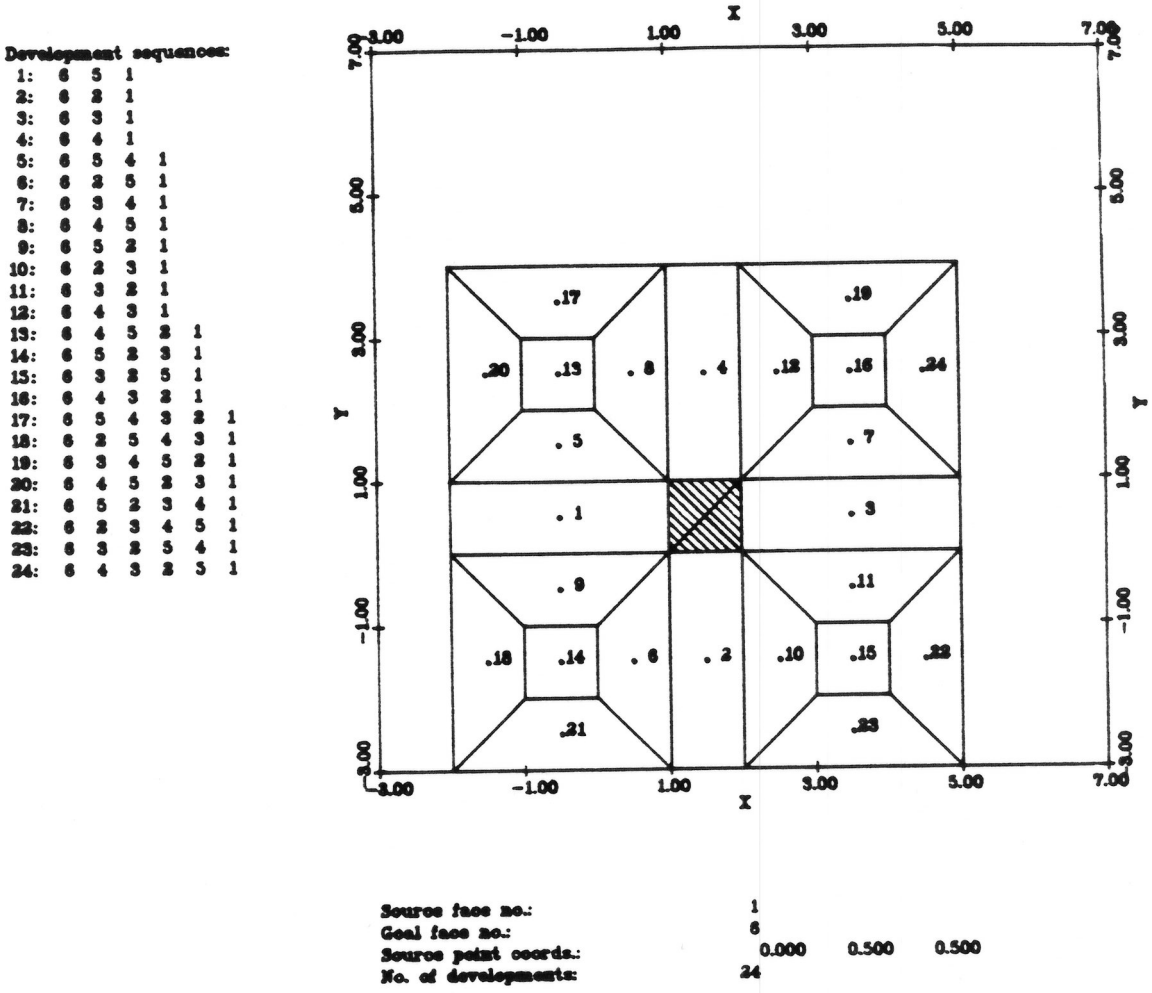
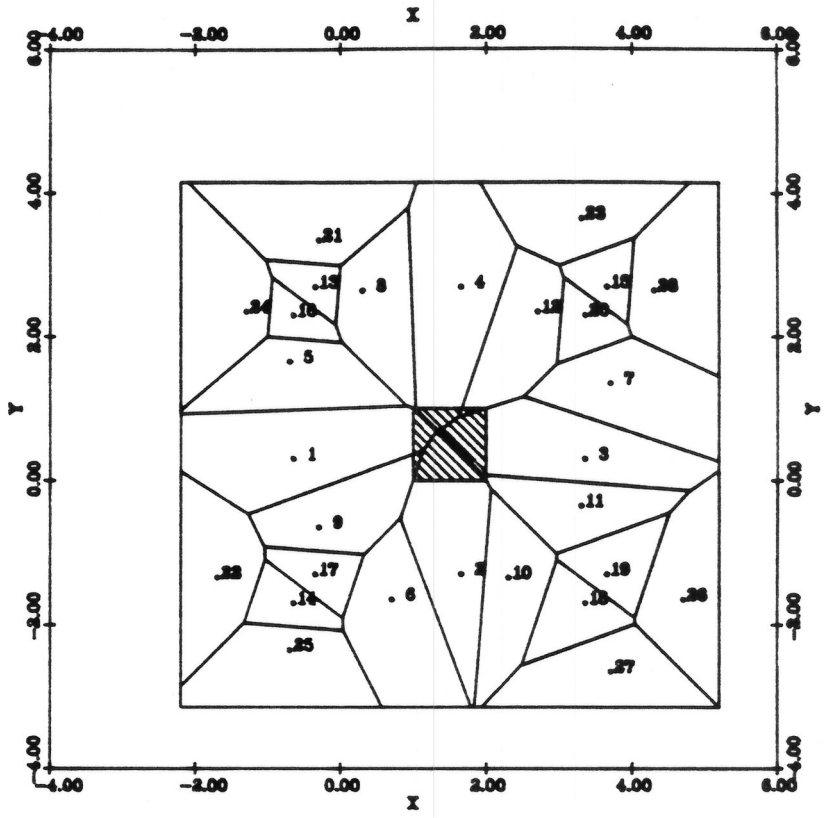


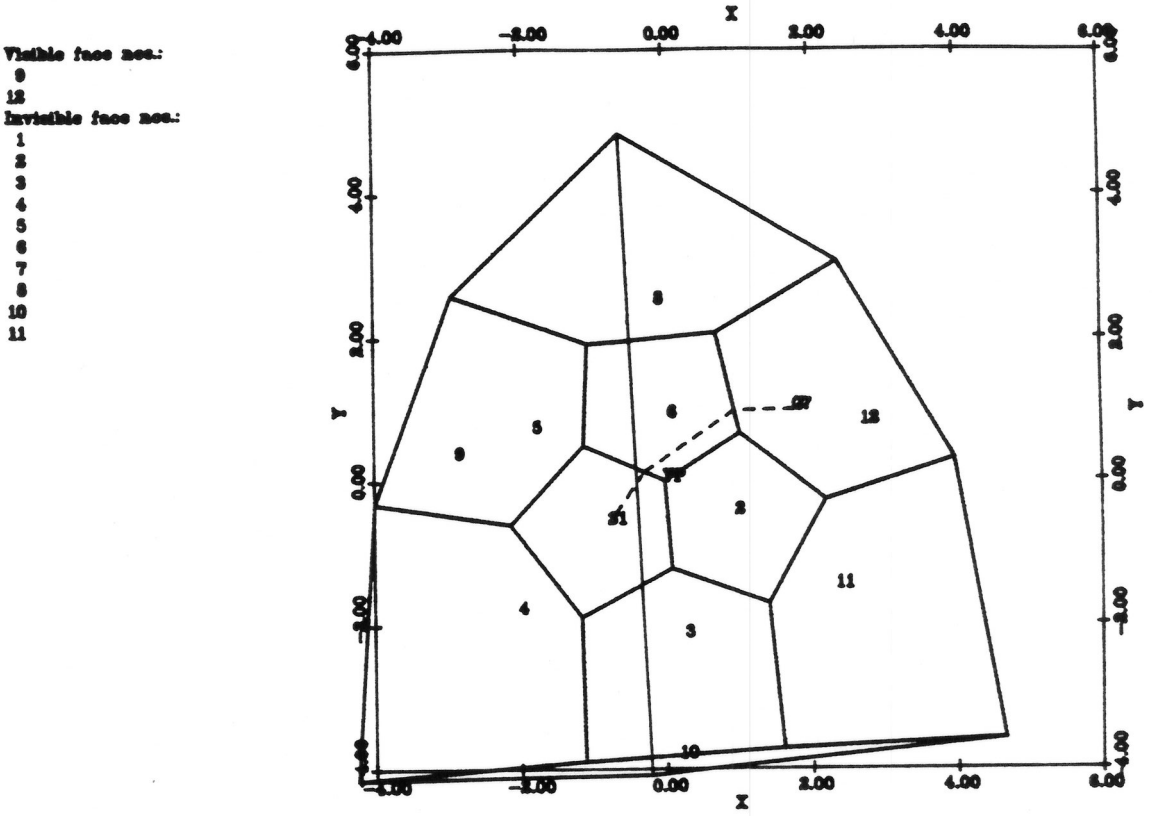
Figure 8 Demonstration of BOUNDARY FINDPATH (locus) on a face of a cube. In figure 8(a) there are 4 regions on the goal face (the shaded polygon) as a result of Voronoi partitioning. Figure 8(b) shows the effect of moving the source on the source face to another location.

Development sequences:

1:	6	5	1			
2:	6	2	1			
3:	6	3	1			
4:	6	4	1			
5:	6	5	4	1		
6:	6	2	5	1		
7:	6	3	4	1		
8:	6	4	5	1		
9:	6	5	2	1		
10:	6	2	3	1		
11:	6	3	2	1		
12:	6	4	3	1		
13:	6	5	4	3	1	
14:	6	2	5	4	1	
15:	6	3	4	5	1	
16:	6	4	5	2	1	
17:	6	5	2	3	1	
18:	6	2	3	4	1	
19:	6	3	2	5	1	
20:	6	4	3	2	1	
21:	6	5	4	3	2	1
22:	6	2	5	4	3	1
23:	6	3	4	5	2	1
24:	6	4	5	2	3	1
25:	6	5	2	3	4	1
26:	6	2	3	4	5	1
27:	6	3	2	5	4	1
28:	6	4	3	2	5	1



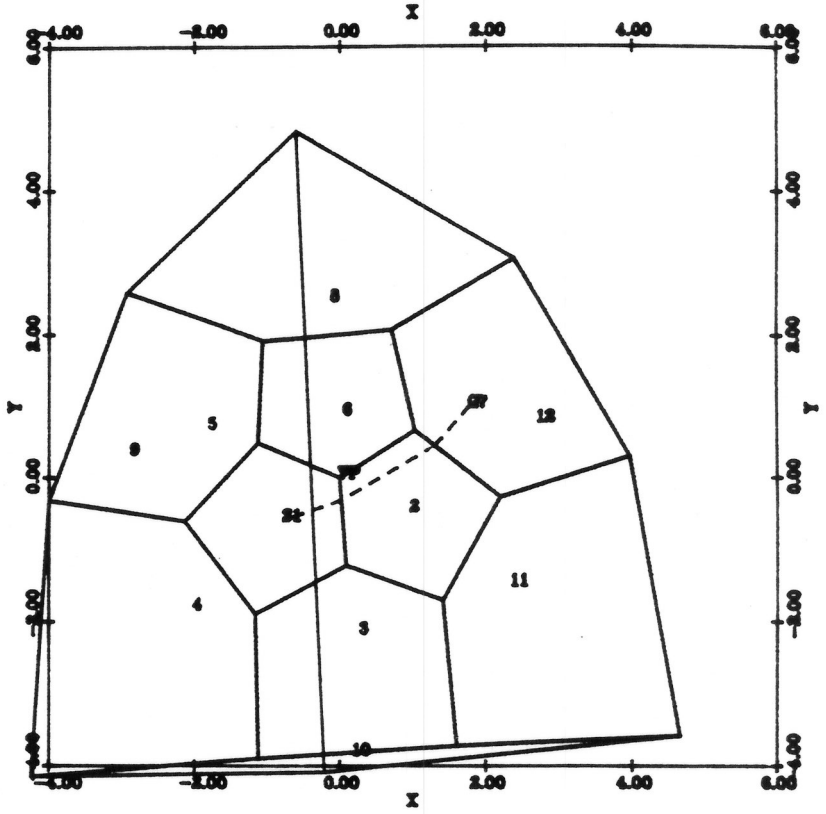
Source face no.: 1
Goal face no.: 6
Source point coords.: 0.000 0.300 0.650
No. of developments: 28



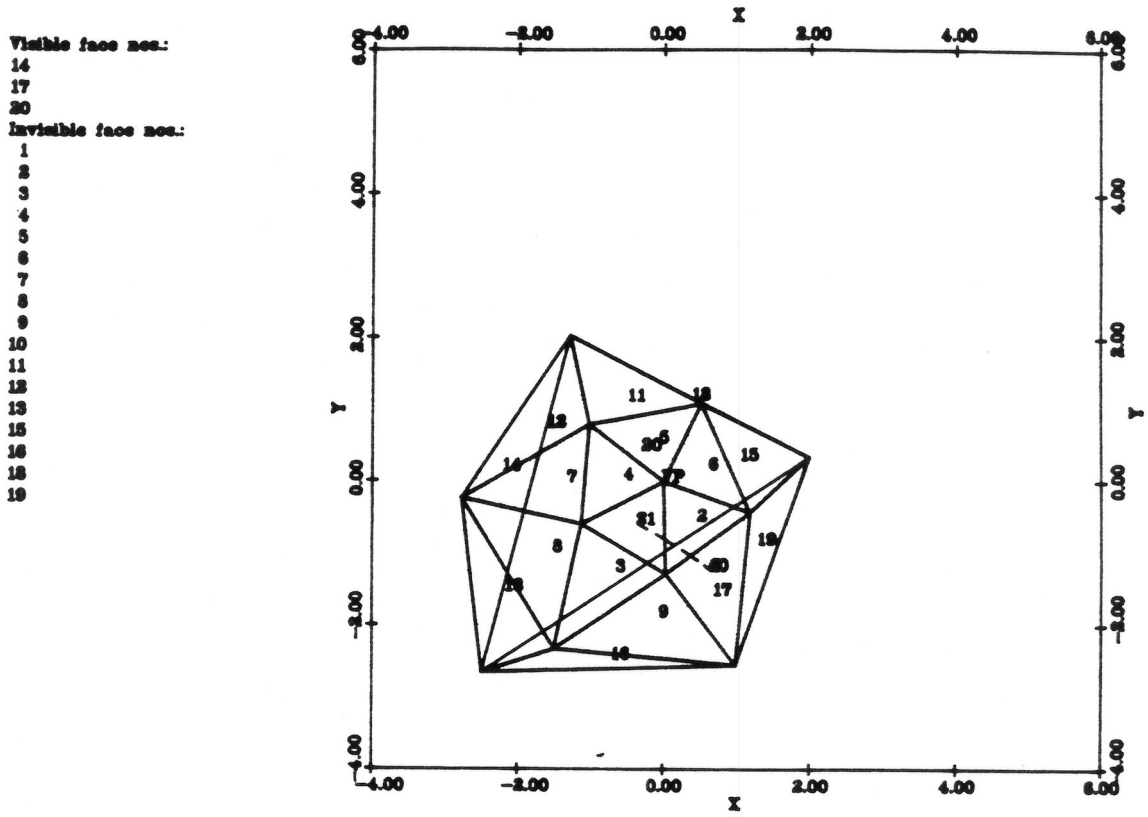
Viewpoint:	1.500	0.800	2.500
Source face no.:	1		
Goal face no.:	7		
Source point:	0.688	0.500	0.000
Goal point:	-0.118	-0.086	1.611
Face development sequence:	1 6 7		
Shortest path length:	2.618		
Shortest path bend points:	0.688	0.500	0.000
	0.263	-0.086	0.000
	-0.236	-0.447	0.986
	-0.118	-0.086	1.611

Figure 9 Shortest paths on the boundary of a dodecahedron.
Both (a) and (b) are shortest paths. This is a perspective view of the object as computed by SP.

Visible face nos.:
9
12
Invisible face nos.:
1
2
3
4
5
6
7
8
10
11



Viewpoint:	1.500	0.800	2.500
Source face no.:	1		
Goal face no.:	7		
Source point:	0.688	0.500	0.000
Goal point:	-0.118	-0.086	1.811
Face development sequence:	1 2 7		
Shortest path length:	2.618		
Shortest path bend points:	0.688	0.500	0.000
	0.000	0.276	0.000
	-0.488	-0.086	0.986
	-0.118	-0.086	1.811



Viewpoint:	0.300	0.800	2.000
Source face no.:	1		
Goal face no.:	10		
Source point:	0.289	0.500	0.000
Goal point:	−0.275	0.936	0.504
Face development sequence:	1 2 10		
Shortest path length:	1.000		
Shortest path bend points:	0.289	0.500	0.000
	0.000	0.867	0.000
	−0.315	0.833	0.193
	−0.275	0.936	0.504

Figure 10 A shortest path on the boundary of an icosahedron. This was computed by SP.

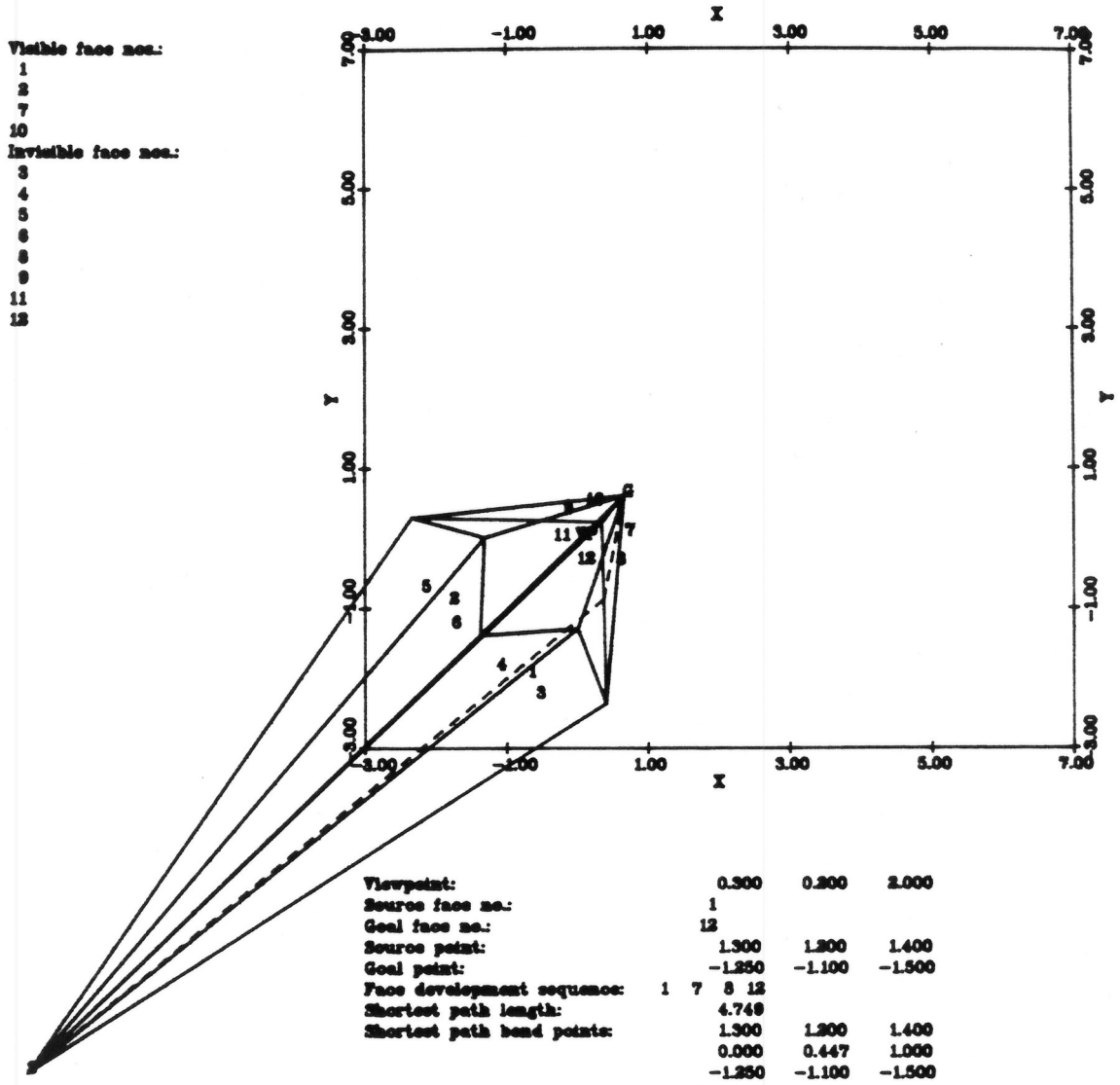


Figure 11 A shortest path around a cube. This was computed by SP after computing the new object and then applying BOUNDARY FINDPATH on it.

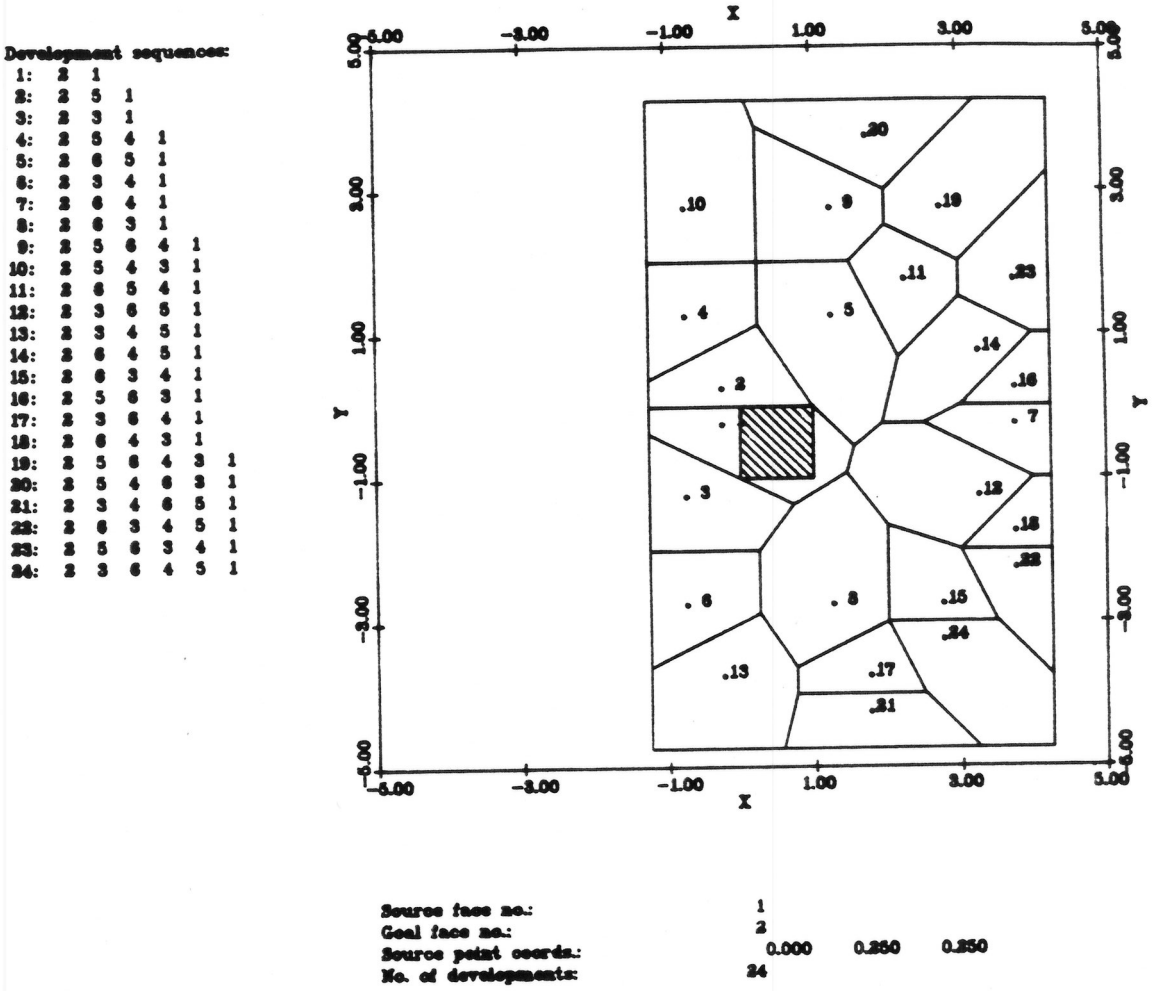
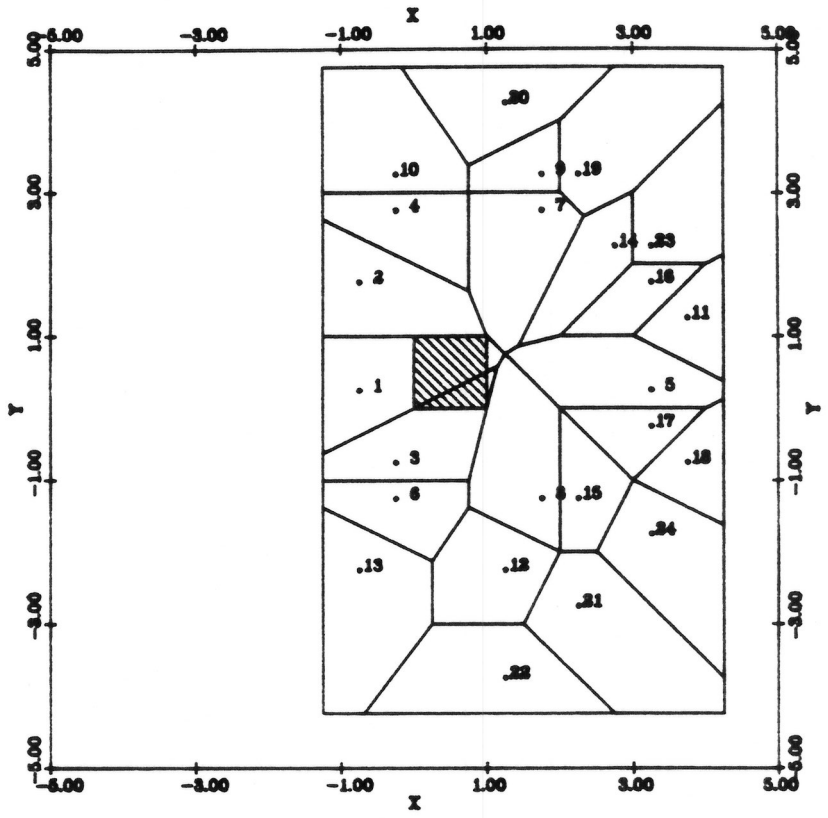


Figure 12 This shows the partitioning of the boundary of a cube in the presence of a source on face 1. Parts (a), (b), (c), (d), and (e) respectively show the regions induced on goal faces 2, 3, 4, 5, and 6. These figures were computed by SP.

Development sequences:

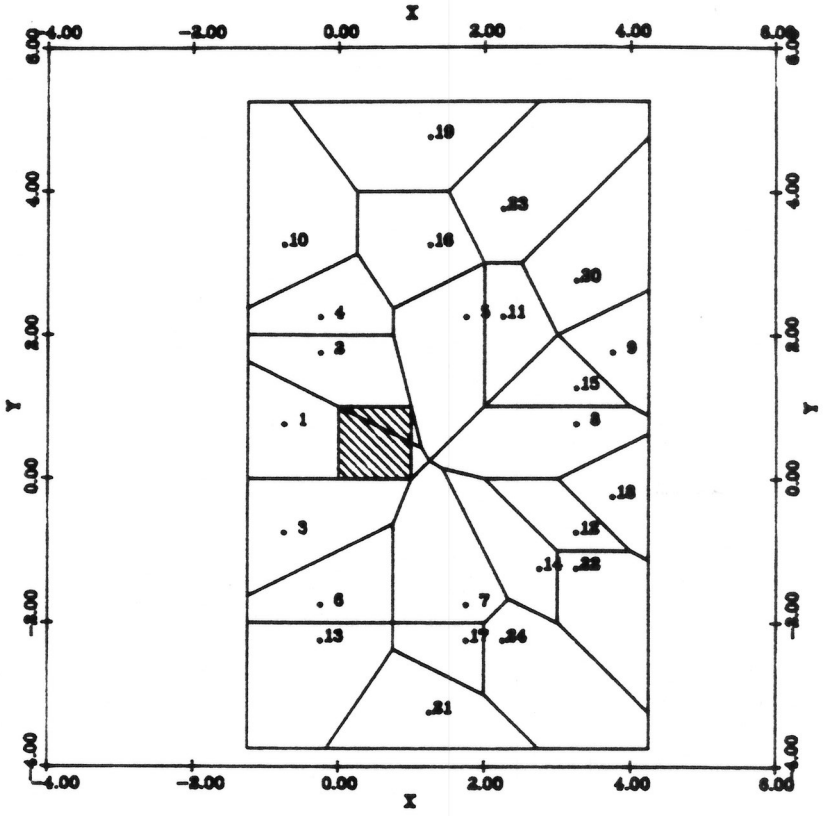
1:	3	1				
2:	3	4	1			
3:	3	2	1			
4:	3	4	5	1		
5:	3	6	5	1		
6:	3	2	5	1		
7:	3	6	4	1		
8:	3	6	2	1		
9:	3	4	6	5	1	
10:	3	4	5	2	1	
11:	3	6	5	4	1	
12:	3	2	6	5	1	
13:	3	2	5	4	1	
14:	3	6	4	5	1	
15:	3	6	2	5	1	
16:	3	4	6	2	1	
17:	3	6	5	2	1	
18:	3	2	6	4	1	
19:	3	4	6	5	2	1
20:	3	4	5	6	2	1
21:	3	2	6	5	4	1
22:	3	2	5	6	4	1
23:	3	4	6	2	5	1
24:	3	2	6	4	5	1



Source face no.: 1
Goal face no.: 3
Source point coords.: 0.000 0.250 0.250
No. of developments: 24

Development sequences:

1:	4	1				
2:	4	5	1			
3:	4	3	1			
4:	4	5	2	1		
5:	4	6	5	1		
6:	4	3	2	1		
7:	4	6	3	1		
8:	4	6	2	1		
9:	4	5	6	3	1	
10:	4	5	2	3	1	
11:	4	6	5	2	1	
12:	4	3	6	5	1	
13:	4	3	2	5	1	
14:	4	6	3	2	1	
15:	4	6	2	5	1	
16:	4	5	6	2	1	
17:	4	3	6	2	1	
18:	4	6	2	3	1	
19:	4	5	2	6	3	1
20:	4	6	5	2	3	1
21:	4	3	2	6	5	1
22:	4	6	3	2	5	1
23:	4	5	6	2	3	1
24:	4	3	6	2	5	1



Source face no.: 1
Goal face no.: 4
Source point coords.: 0.000 0.250 0.250
No. of developments: 24

Development sequences:

1:	5	1				
2:	5	4	1			
3:	5	2	1			
4:	5	4	3	1		
5:	5	6	4	1		
6:	5	2	3	1		
7:	5	6	3	1		
8:	5	6	2	1		
9:	5	4	6	3	1	
10:	5	4	3	2	1	
11:	5	6	4	3	1	
12:	5	2	6	4	1	
13:	5	2	3	4	1	
14:	5	6	3	4	1	
15:	5	6	2	3	1	
16:	5	4	6	2	1	
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18:	5	6	3	2	1	
19:	5	4	6	3	2	1
20:	5	4	3	6	2	1
21:	5	2	3	6	4	1
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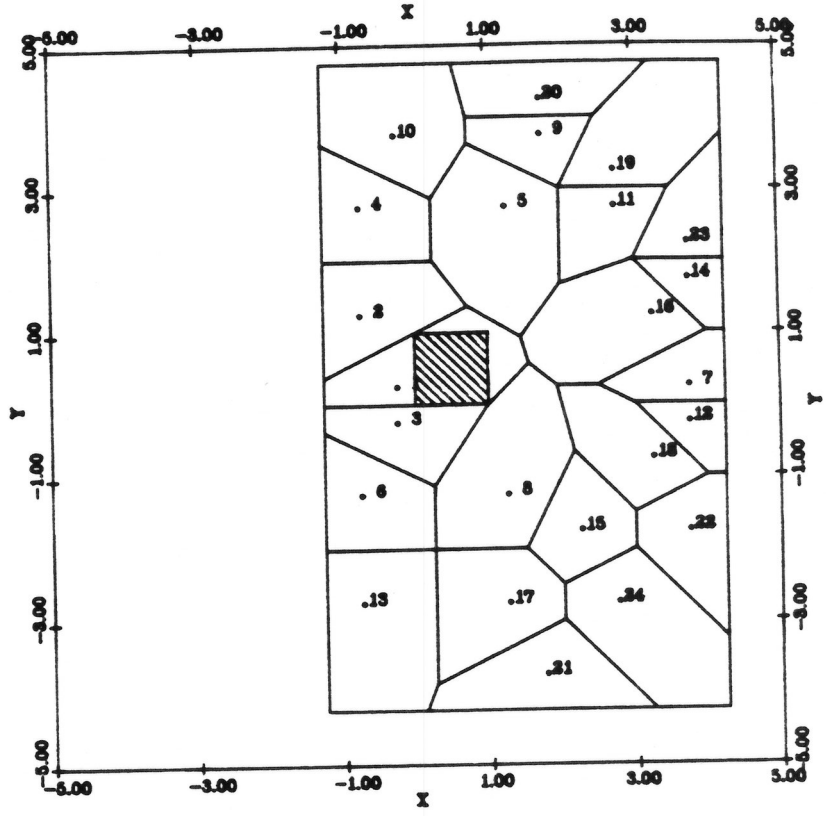
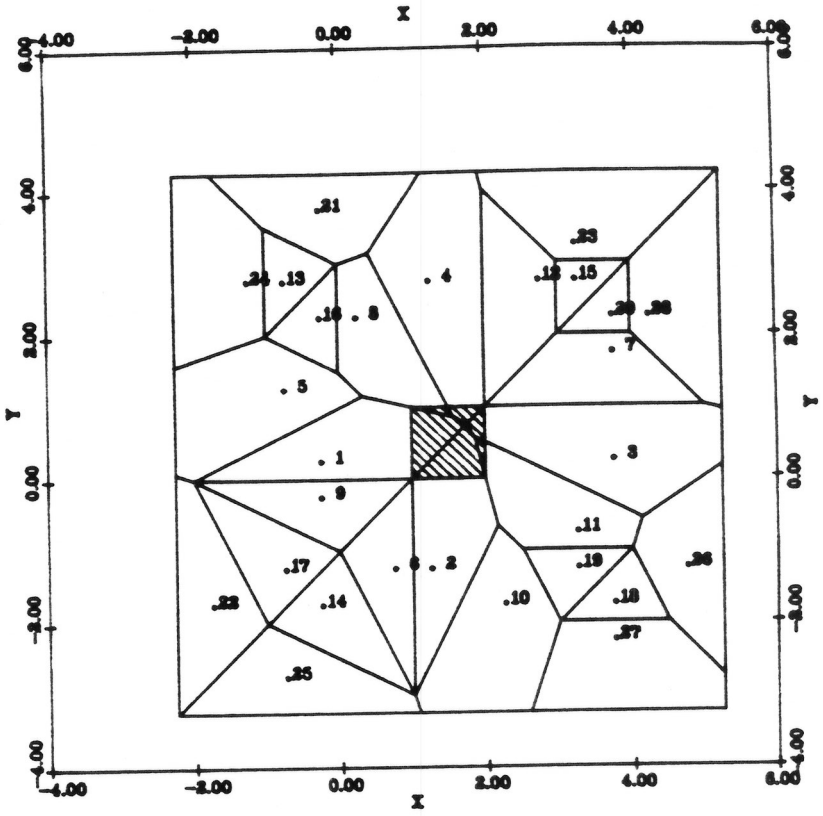
23:	5	4	6	2	3	1
24:	5	2	6	3	4	1



Source face no.: 1
Goal face no.: 5
Source point coords.: 0.000 0.250 0.250
No. of developments: 24

Development sequences:

1:	6	5	1			
2:	6	2	1			
3:	6	3	1			
4:	6	4	1			
5:	6	5	4	1		
6:	6	2	5	1		
7:	6	3	4	1		
8:	6	4	5	1		
9:	6	5	2	1		
10:	6	2	3	1		
11:	6	3	2	1		
12:	6	4	3	1		
13:	6	5	4	3	1	
14:	6	2	5	4	1	
15:	6	3	4	5	1	
16:	6	4	5	2	1	
17:	6	5	2	3	1	
18:	6	2	3	4	1	
19:	6	3	2	5	1	
20:	6	4	3	2	1	
21:	6	5	4	3	2	1
22:	6	2	5	4	3	1
23:	6	3	4	5	2	1
24:	6	4	5	2	3	1
25:	6	5	2	3	4	1
26:	6	2	3	4	5	1
27:	6	3	2	5	4	1
28:	6	4	3	2	5	1



Source face no.: 1
Goal face no.: 6
Source point coords.: 0.000 0.250 0.250
No. of developments: 28