Parallel Volume Computation of Massive Polyhedron Union

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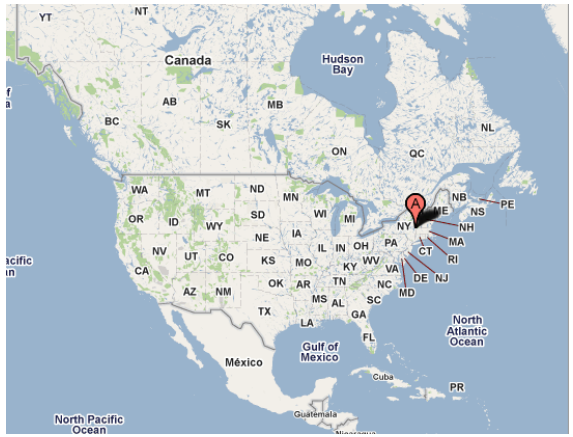

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Large Geometric Datasets vs New HW Capabilities

- Larger geometric datasets $\gg 10^6$ objects
- New parallel HW — restricted capabilities
- $\therefore$ Need new algorithms, data structures.

Why parallel HW?

- More processing $\rightarrow$ faster clock speed
- faster $\rightarrow$ more electrical power
- faster $\rightarrow$ smaller features on chip
- smaller $\rightarrow$ greater electrical resistance !
- $\Longrightarrow\Longleftarrow$.
- Serial processors have hit a wall.

Parallel HW features

- IBM Blue Gene / Intel / NVidia GPU / other.
- Most laptops have NVidia GPUs.
- Thousands of cores / CPUs / GPUs.
- Lower clock speed 750MHz vs 3.4GHz.
- Hierarchy of memory: small/fast $\rightarrow$ big/slow.
- Communication cost $\gg$ computation cost.
- Reads much faster than writes.
- Semaphores protecting parallel writes are very slow.
- Regular memory access much faster than random.
- Efficient for blocks of threads to execute SIMD.
- OS: 187th fastest machine in 6/2013 top500.org runs Windows. 1–186 run Linux variants.

Geometric Databases

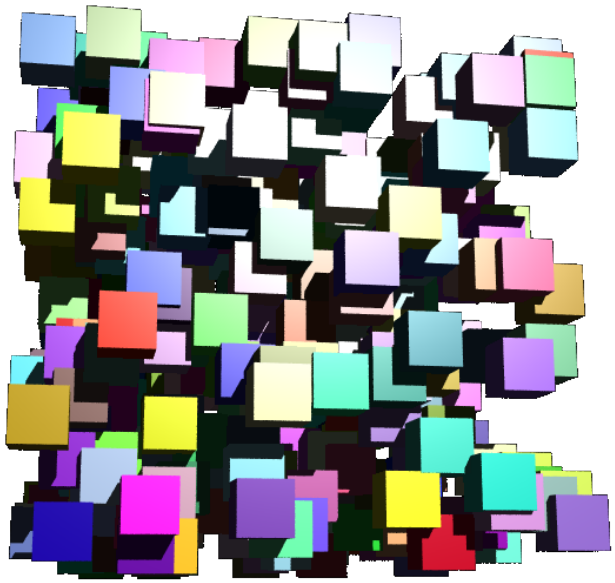
- Larger and larger geometric databases now available, with tens of millions of primitive components.
- Needed operations:
  - interference detection
  - boolean: intersection, union
  - planar graph overlay
  - mass property computation of the results of some boolean operation
- Apps:
  - Volume of an object defined as the union of many overlapping primitives. Two object interfere iff the volume of intersection is positive.
  - Interpolate population data from census tracts to flood zones.

Algorithm Themes

- I/O more limiting than computation $\rightarrow$ minimize storage
- For $N \gg 1000000$, lg N nontrivial $\rightarrow$ deprecate binary trees
- Minimize explicit topology, expecially 3D.
- Plan for 3D; many 2D data structures not easily extensible to 3D, e.g., line sweep.
- E.g., Voronoi diagram: 2D is $\Theta(N \lg N)$. 3D is $\Theta\left(N^2\right)$
- Optimize function composition, e.g. Volume o union.

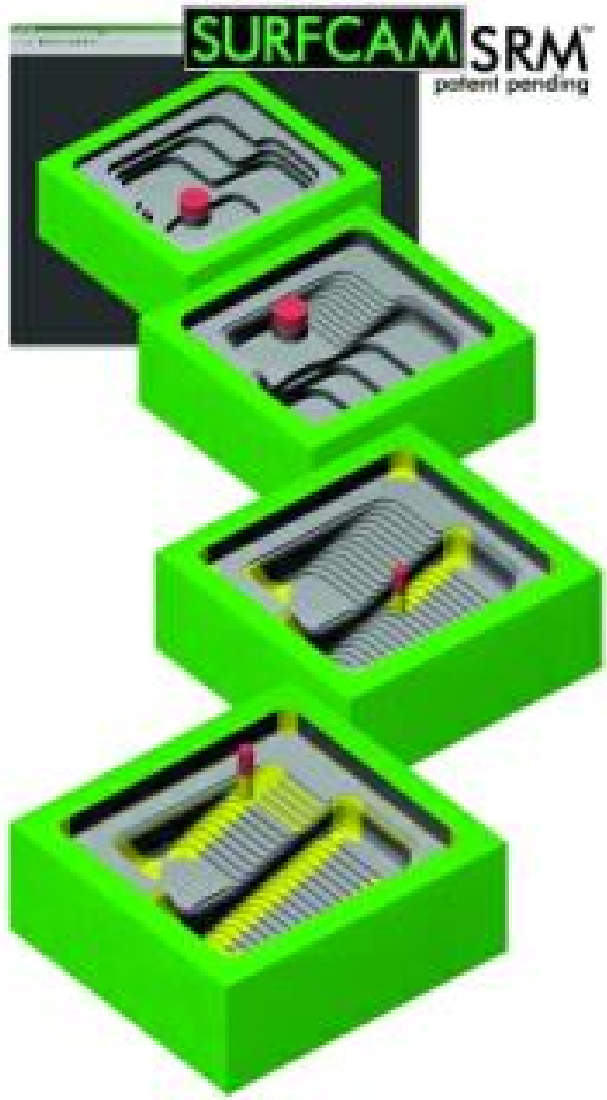
Unifying Example: Volume of Union of Many Cubes

- Nice unifying illustration of several ideas.
- Do a prototype on an easy subcase (congruent axis-aligned cubes).
- Idea extends to general polyhedra.
- **Not** statistical sampling — exact output, apart from roundoff.
- **Not** subdivision-into-voxel method — the cubes' coordinates can be any representable numbers.

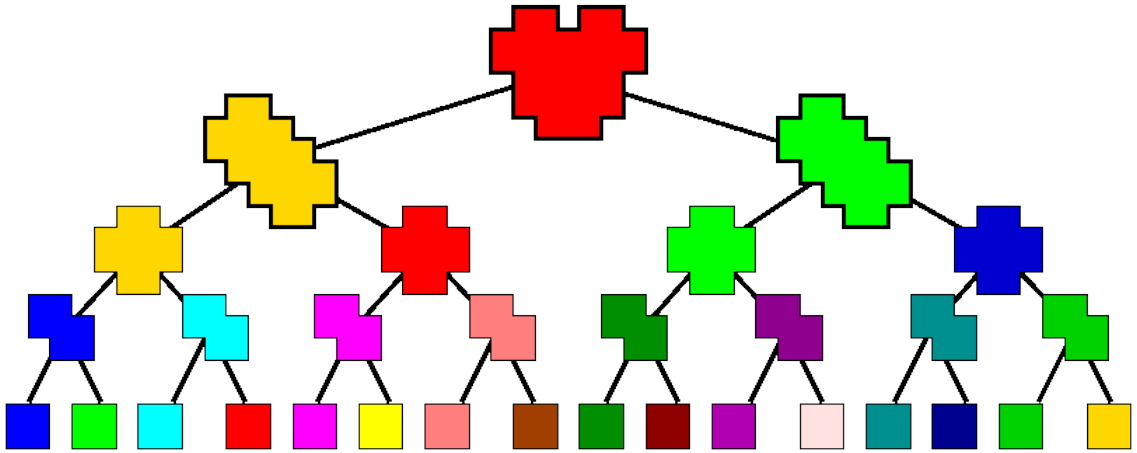


Application: Cutting Tool Path

- Represent path of a tool as piecewise line.
- Each piece sweeps a polyhedron.
- Volume of material removed is (approx) volume of union of those polyhedra.
- Image is from Surfware Inc's Surfcam website.



Traditional N-Polygon Union



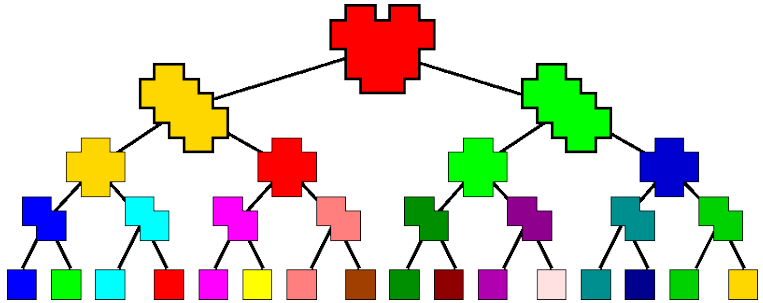
- Construct pairwise unions of primitives.
- Iterate.

Time depends on intermediate swell, and elementary intersection time.

- Let P = size of union of an M-gon and an N-gon. Then P=O(MN).
- Time for union (using line sweep) $T = \Theta(P \lg P)$.
- Total $T = O(N^2 \lg N)$.

Hard to parallelize upper levels of computation tree.

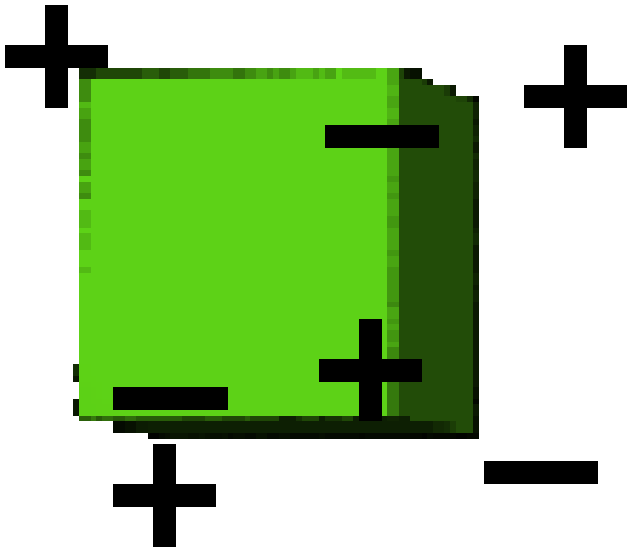
Problems With Traditional Method

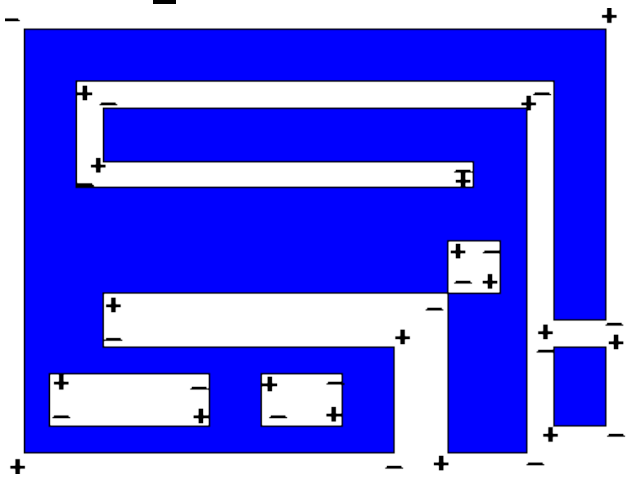


- lg N levels in computation tree cause lg N factor in execution time. Consider $N > 20$.
- Intermediate swell: worse as overlap is worse. Intermediate computations may be much larger than final result.
- The explicit output polyhedron has complicated topology: unknown genus, loops of edges, shells of faces, nonmanifold adjacancies.
- Tricky to get right.
- However explicit output not needed for computing mass properties.
- Set of vertices with neighborhoods suffices.

Volume Determination

Box: $V = \sum_i s_i x_i y_i z_i$
$s_i : +1 or -1$





General rectilinear polygons:

- 8 types of vertices, based on neighborhood
- 4 are type $+$, 4 $-$
- Area = $\sum_i s_i x_i y_i$

- Rectilinear polyhedra: $V = \sum_i s_i x_i y_i z_i$
- $\exists$ formulae for general polyhedra.

Properties

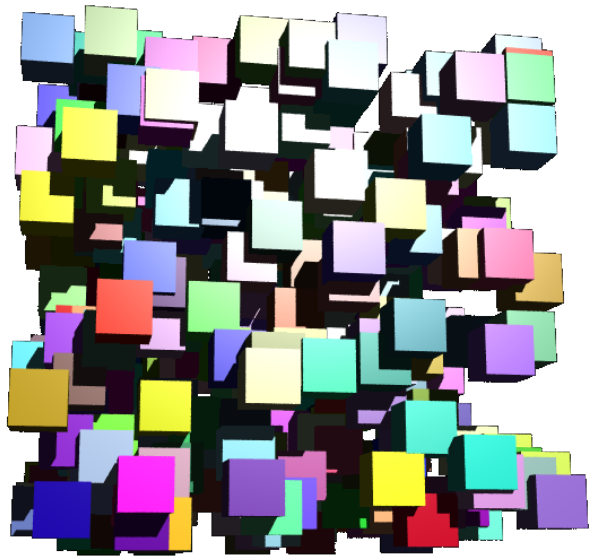
Represent output union polyhedron as set of vertices with neighborhoods.

- no explicit edges; no edge loops.
- no explicit faces; no face shells.
- no component containment info.
- general polygons ok: multiple nested or separate comps.
- any mass property determinable in one pass thru the set.
- parallelizable.
- This is very useful because computing only the output vertices and neighborhoods is much faster than also computing the edges, faces, and global topology.

Volume Computation Overview

- Find all vertices of output object.
- For each vertex, find location and local geometry.
- Map-reduce sum over vertices, applying formula.

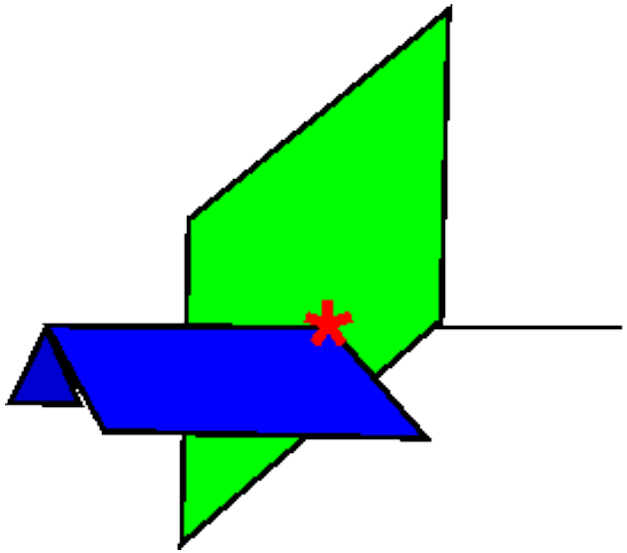
Challenge: to find those vertices in expected constant time per vertex.



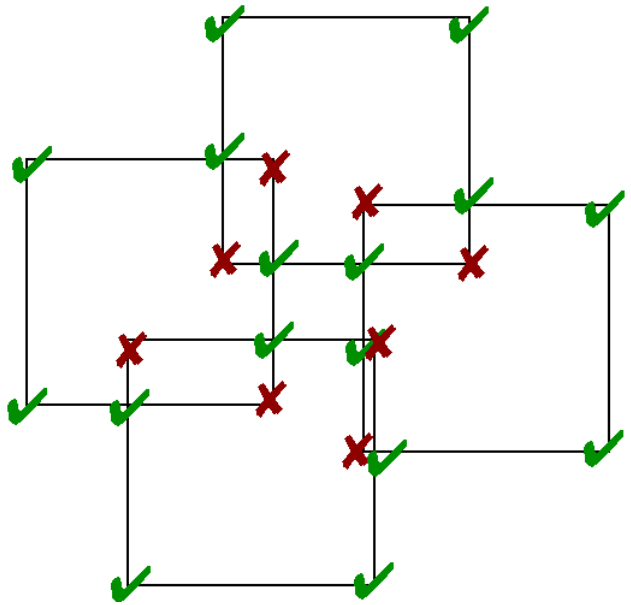
Finding the Vertices

3 types of output vertex:

- Input vertex,
- Edge–face intersection,
- Face–face–face intersection.



- Find possible output vertices, and filter.
- An output vertex must not be contained in any input cube.
- Isn't intersecting all triples of faces, then testing each candidate output vertex against every input cube too slow?
- No, if we do it right.



3D Uniform Grid

Summary

- Overlay a uniform 3D grid on the universe.
- For each input primitive — cube, face, edge — find overlapping cells.
- In each cell, store set of overlapping primitives.

Properties

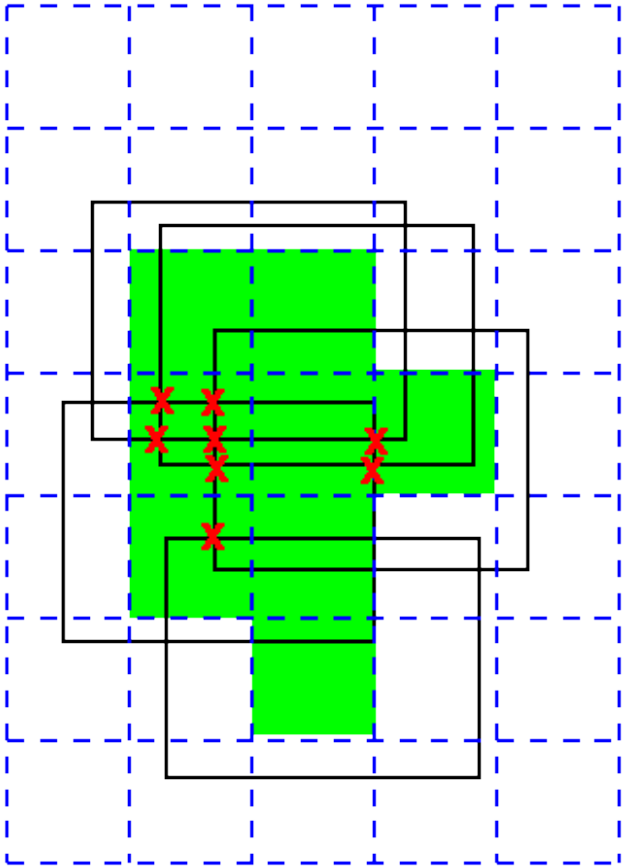
- Simple, sparse, uses little memory if well programmed.
- Parallelizable.
- Robust against moderate data nonuniformities.
- Bad worst-case performance on extremely nonuniform data.
- Ditto any hierarchical method like octree.

Advantage

- Intersecting primitives must occupy the same cell.
- The grid filters the set of possible intersections.

Adding the Cubes Themselves to the Grid

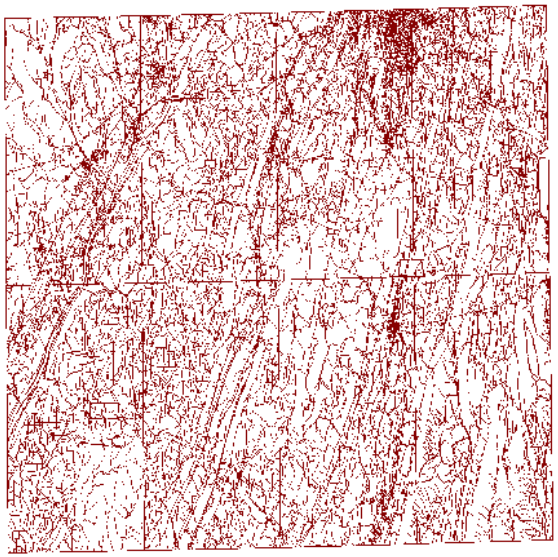
- For each cube, find cells it completely covers.
- When a cell is completely covered by a cube: nothing in that cube can contribute to the output. So:
- Find covered cells first.
- Do not insert objects into covered cells.
- Intersect pairs and triples of objects in non-covered cells.



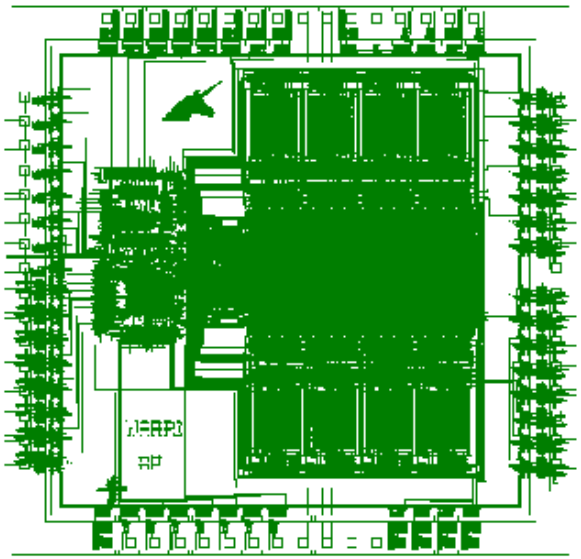
For cell size somewhat smaller than edge size, almost no hidden intersections found. Good.
Expected time = Θ(size(input) + size(useful intersections)).

Uniform Grid Qualities

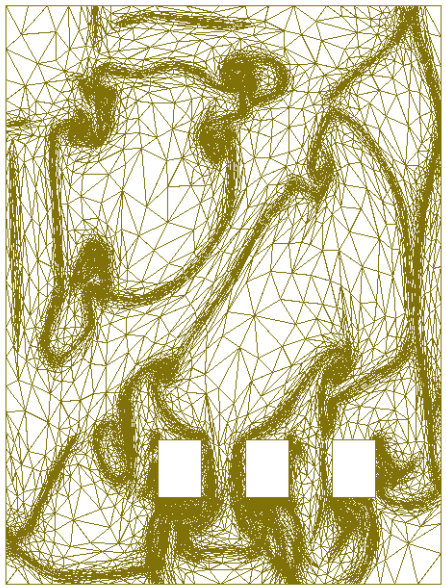
- **Major disadvantage:** It's so simple that it apparently cannot work, especially for nonuniform data.
- **Major advantage:** For the operations I want to do (intersection, containment, etc), it works very well for any real data I've ever tried.



USGS Digital Line Graph



VLSI Design



Mesh

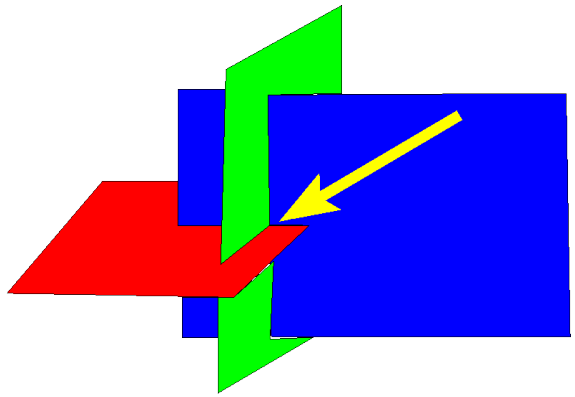
Uniform Grid Time Analysis

Show that time to find edge–edge intersections in E^2 is linear in input+output size regardless of varying number of edges per cell.

- N edges, length 1/L, $G \times G$ grid, η edges per cell.
- $\overline{\eta} = \lambda_\eta \stackrel{\Delta}{=} \frac{N}{G^2}(G/L + 1)$
- Poisson distribution, parameter λ_η.
- Expected number of edge–edge tests: $\overline{(\eta^2 - \eta)}$
- $\overline{\eta} = \lambda_\eta$ and $\overline{\eta^2} = {\lambda_\eta}^2 + \lambda_\eta$.
- Expected number of intersection tests per cell: ${\lambda_\eta}^2 = \frac{N^2}{G^4}(G/L + 1)^2$
- Expected total number of intersection tests, over the G^2 cells: $\frac{N^2}{G^2}(G/L + 1)^2$.
- Total time: insert edges into cells + test for intersections $T = \Theta\left(N(LG + 1) + \frac{N^2}{G^2}(G/L + 1)^2\right)$.
- Minimized when $G = \Theta(L)$, giving $T = \Theta\left(N + N^2 L^{-2}\right)$.
- Q.E.D.

Face–Face–Face Intersection Details

- Iterate over grid cells.
- In each cell, test all triples of faces, each from a different cube.
- Three faces intersect if their planes intersect, and the intersection is inside each face (2D point containment).
- Then look up s_i in a table and update accumulating volume.
- Cubes are easier than general polyhedra.



Point Containment Testing

Question:

- P is a possible vertex of the output union polyhedron.
- Is point P contained in any input cube?

Answer:

- Find which cell, C, (if any) contains P.
- If C is completely covered by some cube then P is inside the covering cube.
- Otherwise, test P against all the cubes that overlap C.
- Expected number of such cubes is constant, under broad conditions.
- Expect test time per P: constant.

Face–Face-Face Intersection Execution Time

- N: number of cubes; $1/L$: edge length; $1 \times 1 \times 1$ universe.
- Expected number of 3-face intersections = $\Theta\left(N^3 L^{-6}\right)$.

Effect of Grid

- Choose G: number of grid cells on a side $= 2L$.
- Number of face triples: N^3
- Prob. of a 3-face test succeeding = $N^{-2}L^{-6}$.
- Depending on asymptotic behavior of L(N), this tends to 0.
- Prob. of 3 tested faces actually intersecting = c, indep. of N and L(N).
- Big improvement!

Effect of Covered Cells

- Expected number of 3-face intersections = $\Theta(N^3 L^{-6})$.
- However, for uniform i.i.d. input, expected visible number: $\Theta(N)$.
- Prob. computed intersection is visible = c, indep. of N and L(N).
- Time to test if a point is inside any cube also constant.
- Total time reduces to $\Theta(N)$.

Parallel Implementation

- 32 thread OpenMP.
- Dual 3.4GHx Xeon, 128GB memory.
- Don’t slice up the input spatially.
- Inserting objects into cells uses atomic increment and capture.
- Very compact data structures.
- Compute element-cell incidences twice: 1st time just to count size of each cell.
- Then allocate ragged array.
- Finally compute again and populate array.
- Intersection testing and volume computation writes only with sum reduce.
- 824 executable lines of C++.

Results

- Fast small datasets: **N=1000**, L=10: V=0.573, A=19, **T=0.02s**.
- Medium: **N=1,000,000**, L=100, $G = 200$: V=0.6, **T=3.3s**.
- Feasible large datasets: **N=100,000,000**, L=400, G=1000: V=0.765, A=831, 57GB, **T=473s**.
- **10x faster with 32 threads than with one thread.**
- 258,461,149 of the 1,000,000,000 grid cells were completely covered by some cube.
- Smaller cubes are faster: **N=100,000,000**, L=500, G=1000: **T=396s**, 50GB.

Output vertex types for big case:

- 186,366,729 of the 800,000,000 input vertices outside all cubes.
- 395,497,686 of the intersections of 3 of the 600,000,000 input faces.
- 811,328,383 of the intersections of one of the 600,000,000 input faces with one of the 1,200,000,000 input edges.

Implementation Validation

~~It compiles and runs w/o crashing; why look for trouble?~~

- Terms summed for volume are large and mostly cancel.
- Errors unlikely to total to a number in [0,1].
- Expected volume: $1 - \left(1 - L^3\right)^N$.
- Compare to computed volume.
- Assume that coincidental equivalence is unlikely.
- Construct specific, maybe degenerate, examples with known volume.

Extensions

To general boolean ops:

- Intersection of many convex polyhedra quite easy.
- Any boolean op expressible in CNF as union of intersections (common technique in logic design for computer HW).

To general polyhedra:

- Formulae are messier.
- Roundoff error would be biggest problem.
- Fatal to miss an intersection.
- Compute using rationals, perhaps with GMPXX.
- Time cost: factor of 100?

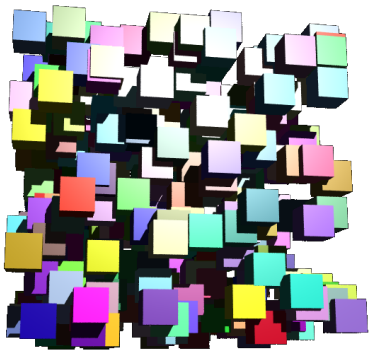
Testing polyhedron validity: Illegal volume or volume change after rigid transformation $\rightarrow$ invalid.

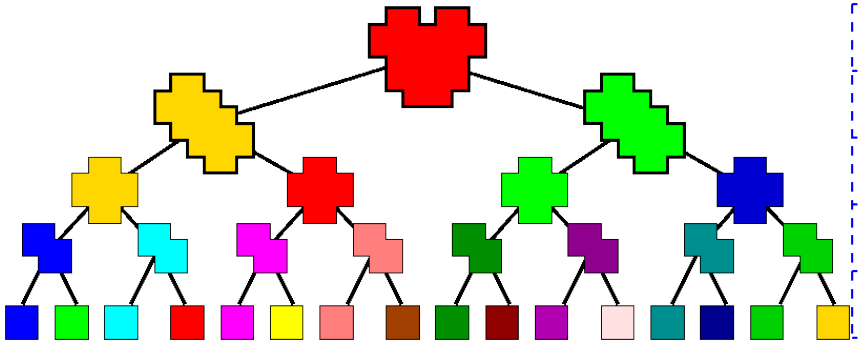
Summary — To Process Big Geometric Datasets on Parallel Machines

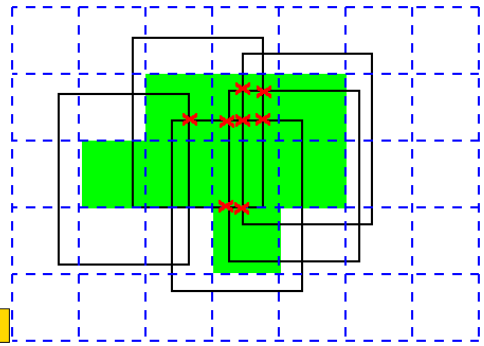
Guiding principles:

- Use minimal possible topology, and compact data structures.
- Short circuit the evaluation of volume(union(cubes)).
- I/O sensitive: design for expected input.
- Use lots of memory; run BIG examples to show the linear time.

Allows very large 3-D datasets to be processed quickly.







Source code (prototype quality) freely available for nonprofit research and education; I welcome stress tests and error reports.