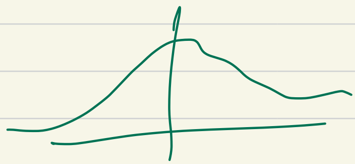
PROBABILITY CLASS 21 R 3/31/22

EXAM MIGHT HAVE STUFF FROM EARLY IN SEMESTER.

GAUSSIAN $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ $\quad -\infty < x < \infty$

$\mu = 0 \quad \sigma = 1$



MOST OTHER DISTRIBUTIONS START TO LOOK LIKE THIS FOR LARGE N

TO PROVE $\int f(x)\,dx = 1$

LET $A = \int \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$

$$= \int \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy$$

← MULTIPLY THEM

$$A^2 = \frac{1}{2\pi}\iint e^{-\left(\frac{x^2+y^2}{2}\right)}\,dy\,dx$$

CHANGE VARIABLES $\quad x = R\cos\theta$

$0 \le R < \infty \quad y = R\sin\theta$

$0 \le \theta < 2\pi$

$$J = \frac{d(x,y)}{d(R\theta)} = \begin{vmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{vmatrix}$$

$$= R(\cos^2 + \sin^2) = R.$$

$$dx\,dy = R\,dR\,d\theta.$$

$$A^2 = \frac{1}{2\pi}\iint R e^{-\frac{R^2}{2}}\,dR\,d\theta$$

$$= \int R e^{-\frac{R^2}{2}}\,dR$$

$$= e^{-R^2/2}\Big|_0^\infty = 1$$

$A^2 = 1 \qquad A = 1$ $\quad$ THIS IS A LEGAL PDF.

I CONVERTED FROM CARTESIAN TO POLAR SO NOW I CAN INTEGRATE IT.

~~2 VARIABLE~~

$$\int_0^{2\pi} d\theta = 2\pi$$

2 VARIABLE GAUSSIAN

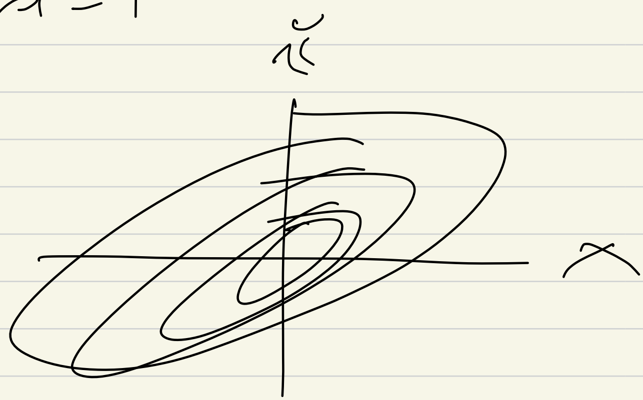
$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\left(\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right)}$$

ρ : CORRELATION COEFFICENT

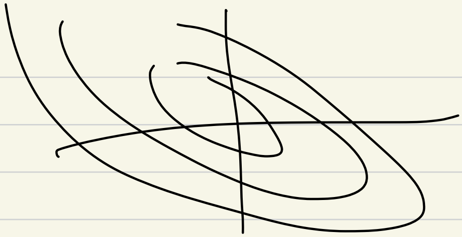
$|\rho| \leq 1$

CONTOUR PLOT

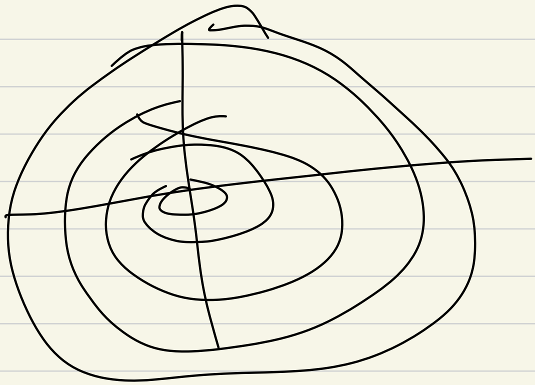
$\rho > 0$



$\rho < 0$



$\rho = 0$



I'M SHOWING YOU SPECIFIC RESULTS AND GENERAL TECHNIQUES,

PROOF

$$\int_{-\infty}^{\infty} f(x,y)\,dy = f(x)$$

ONE CAN IMAGINE MANY WAYS TO HAVE A 2 VARIABLE P.D.F WITH GAUSSIANS. HERE A SPECIFIC USEFUL ONE.

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\left(\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right)}$$

$$\boxed{\begin{array}{c}\text{EQN } 5.18 \\ \text{p} 253\end{array}}$$

SHOW $\int f(x,y)\,dy = 1$

$$\boxed{\$ = \frac{e^{\frac{-x^2}{2(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} \int e^{-\left(\frac{-2\rho xy+y^2}{2(1-\rho^2)}\right)}\,dy}$$

COMPLETE SQUARE ON

$\rho^2 x^2 - 2\rho xy + y^2 - \rho^2 x^2$

$(\rho x - y)^2 - \rho^2 x^2$

$$\beta = \frac{e^{\frac{-x^2+\rho^2 x^2}{2(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}} \underbrace{\int \underbrace{e^{-\frac{y^2}{2(1-\rho^2)}}}_{e^{-\frac{w^2}{2}}} dy}$$

$$w = \frac{y}{\sqrt{1-\rho^2}} \qquad dy = dw\sqrt{1-\rho^2}$$

$$\underbrace{\int \sqrt{1-\rho^2}\, e^{-\frac{w^2}{2}} dw}_{\sqrt{1-\rho^2}}$$

$$\beta = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

(THERE MAY BE A MISS(N) $\sqrt{2\pi}$)

$$\underline{\int f(x,y)\, dy = f(x)}$$

$$f(y|x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{\frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{(x^2-2\rho xy+y^2)}{2(1-\rho^2)}}}{\frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}}$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)}}\, e^{\left[C\right]}$$

$$C = \frac{x^2}{2} - \frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}$$

$$= \frac{\cancel{x^2} - x^2\rho^2 - \cancel{x^2} + 2\rho xy - y^2}{2(1-\rho^2)}$$

$$= -\frac{(\rho^2 x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}$$

$$f(y|x) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\left(\frac{\rho^2 x^2 - 2\rho x y + y^2}{2(1-\rho^2)}\right)}$$

$$= (\rho x - y)^2$$

$$f(y|x) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{(\rho x - y)^2}{2(1-\rho^2)}}$$

HOW IS THIS USEFUL.

WE HAVE A NOISY COMM CHANNEL.

x IN, y OUT.

WE SEE y, WHAT WAS x?

THERE ARE DIFFERENT WAYS DEPENDING ON WHAT WE HAVE.

HERE, $f(y|x)$ USED $f(x)$.

ML = MAX LIKELIHOOD METHOD

GIVEN $f(y|x)$ FIND x TO MAX THAT.

THAT IS $\rho x - y = 0$ $\quad x = \frac{y}{\rho}$

BEST x IS $\frac{y}{\rho}$.

$\boxed{\text{EX 6.26 p 323}}$

MAP = MAX A POSTERIORI METHOD

$\max_x f(x|y)$

REVIEW CONDITIONAL EXPECTATION

p 268

HAVE X, Y $\quad E[Y|X]$

TO SHOW $\boxed{E[Y] = E\left[E[Y|X]\right]}$

$$E[Y] = E[E[Y|X]]$$

GIVEN $f_X(x)$ $f_Y(y|x)$

$$E[X] = \int x f(x)\,dx$$

$$E[g(X)] = \int g(x) f(x)\,dx$$

$$E[E[Y|X]] = \int E[y|x] f(x)\,dx$$

$$= \int y f(y|x)\,dy$$

$$\iint \underbrace{y f_Y(y|x) f(x)}_{f(x,y)}\,dy\,dx$$

$$= \int \underbrace{\left(\int f(x,y)\,dx \right)}_{f(y)} y\,dy$$

$$= \int y f(y)\, dy = E[Y]$$

$$E[E[Y|X]] = E[Y].$$

CONDITIONAL EXPECTATION

BACK TO P 334

ESTIMATORS.

IN : X   OUT : Y

WE SEE Y, WHAT WAS X?

WE SAW SOME LINEAR ESTIMATORS.

MAP, ML. $x = cy$. FOR SOME c.

NOW MORE POWERFUL ESTIMATOR

NONLINEAR.

MIN MEAN SQUARE ERROR (MSE)

ESTIMATOR.

OUR GUESS FOR x WILL BE A FUNCTION OF y : $g(y)$

WHAT SHOULD g BE?

OUR COMPUTED X IS $\hat{X}$

SEE Y, COMPUTE $\hat{X}$

ERROR $X - \hat{X}$

SQUARED $(X - \hat{X})^2$ $\hat{X} = g(Y)$

$E[(X - g(Y))^2]$

WANT TO PICK g TO MIN $E[-]$.

WHY SQUARE? IT MAKES MATH EASIER.

$X, Y \quad g(Y) \quad \min_g E[(X-g(Y))^2]$

REALLY SIMPLE $g(Y)$

MAKE IT CONSTANT $\quad g(Y) = a$

FOR SOME a.

WHAT'S BEST a?

BEST a MIN $E[(X-a)^2]$

$= E[X^2 - 2aX + a^2]$

$= E[X^2] - 2aE[X] + a^2$

$= E[X^2] - (E[X])^2 + \underbrace{E[X]^2 - 2aE[X] + a^2}_{(E[X]-a)^2}$

MIN AT $a = E[X]$.

NOW A LITTLE MORE COMPLICATED

ALLOW $g(Y) = aY + b$

LINEAR WHAT ARE BEST a, b?

$$\min_{a,b} \left((X - (aY + b))^2 \right)$$

WHAT'S BEST b GIVEN EVERYTHING ELSE?

$$b = E[X - aY]$$
$$= E[X] - a\,E[Y].$$

$$\text{ERR} = (X - aY - E[X] + a\,E[Y])^2$$
$$= (X - E[X] + a(E[Y] - Y))^2$$

VARY a TO MIN THAT...

QUADRATIC IN a SO DERIVATIVE $= 0$

$\frac{d}{da}$: $\quad a^* = \rho_{XY} \dfrac{\sigma_X}{\sigma_Y}$

WHERE $\sigma_x = E[x^2] - E[x]^2$

ETC.

$\boxed{\text{TOP OF P335}}$