PROB C20 M 3/28/22

6.7 3.8
6.9 3.10
6.10
6.11
5.31 3.18
5.35 3.2

EX 6.7

X_1 $U[0,1]$

X_2 $U[0,x_1]$

X_3 $U[0,x_2]$

FIND JOINT PDF OF $X_1X_2X_3$
PDF OF X_3

$X_1 : U[0,1]$ $\qquad f_{X_1}(x_1) = 1$

$0 \le x_1 \le 1$

$X_2 : U[0, X_1]$

$$f_{X_2}(x_2 | x_1) = \begin{cases} \dfrac{1}{x_1} & 0 \le x_2 \le x_1 \\ 0 & \end{cases}$$

eg $x_1 = \frac{1}{2}$ $\qquad f(x_2 | x_1 = \tfrac{1}{2}) = \begin{cases} 2 & \text{if } 0 \le x_2 \le \frac{1}{2} \\ 0 & \end{cases}$

$X_3 \;\; U[0, x_2]$

$$f(x_3 | x_2) = \frac{1}{x_2} \qquad 0 \le x_3 \le x_2$$

$$f(x_1, x_2, x_3) = f(x_3 | x_2)\, f(x_2)$$

$$= f(x_3 | x_2)\, f(x_2 | x_1)\, f(x_1)$$

$$f(x_1 x_2 x_3) = \frac{1}{x_2 x_1} \qquad -x_1$$

$$0 \le x_3 \le x_2 \le x_1 \le 1$$

$$f(x_3) = \iint f(x_1 x_2 x_3)\, dx_1 dx_2$$

$$= \int_0^1 \left[\int_0^{x_1} \frac{1}{x_1 x_2}\, dx_2\right] dx_1$$

$$= \int \underbrace{\frac{1}{x_1} \underbrace{\left[\int_0^{x_1} \frac{1}{x_2}\, dx_2\right]}_{-\frac{\ln x_2}{x_2}}}\, dx_1$$

$$f(x_3) = \frac{1}{2}(\ln x_3)^2$$

EX 6.9 p 310

RV $X_1 \cdots X_N$

$W = MAX(X_i)$

$Z = MIN(X_i)$

X_i INDEPENDENT I.I.D.

HAVE $F_{X_1}(X_1)$ WANT $F_W(w)$

$F_W(w) = PROB(W \leq w)$

$= PROB(W \leq MAX(X_1 X_2 X_3 \cdots))$

$= PROB[W \leq X_1 \; \& \; W \leq X_2 \; \& \; W \leq X_3 \cdots]$

$= P[W \leq X_1] \, P[W \leq X_2] \cdots P[W \leq X_N]$

$= F_{X_1}(w) \, F_{X_2}(w) \cdots$

$= F_{X_1}(w)^N$

$X_i \quad U[0,1] \quad F(x) = x$

$w = MAX(X_i)$

$F(w) = w^n$

2 RV $\quad F(w) = w^2 \quad f(w) = 2w$

3 RV $\quad F(w) = w^3 \quad 3w^2$

DENSITY IS GREATER FOR LARGER w. $\quad w = max(X_i)$

FOR $Z = MIN(X_i)$

$1 - F_Z(z) = P[Z \geq z]$

$Z = MIN(X_i)$

$P[Z \geq z] = P[MIN(X_i) \geq z]$

$= P[X_1 \geq z]\,P[X_2 \geq z] \ldots$

$$1 - F_Z(z) = \{1 - F_X(z)\}^n$$

$X_i \sim U[0,1]$ $\quad F(x_i) = x_i$

$1 - F(x_i) = 1 - x_i$

$z = \min(X_i)$

$n = 2$ $\quad 1 - F(z) = (1 - z)^2$

$= 1 - 2z + z^2$

$$\underline{F(z) = 2z - z^2}$$

YOU COULD ALSO LET

W = MIDDLE OF X_1, X_2, X_3

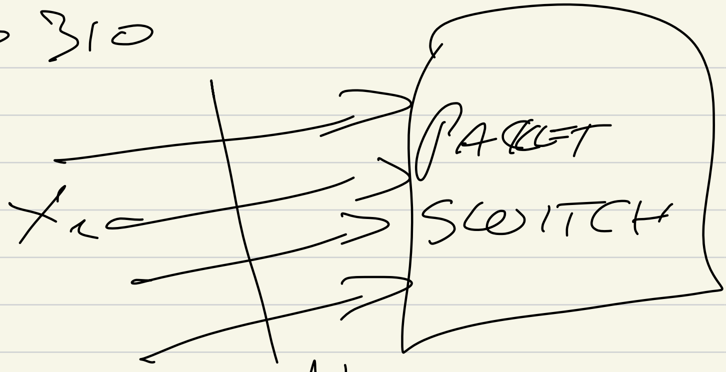
FIND $f(w)$ . .

EVEN IF X_i ARE $U[0,1]$,

W WILL BE CONCENTRATED IN

MIDDLE.

6.10 P 310



N INDEP SOURCES

EACH SOURCE X_n:

INTERARRIVAL TIME EXPONENTIAL
RATE λ_n

MEAN INTERARRIVAL TIME $\frac{1}{\lambda_n}$.

WE WANT TO KNOW
INTERARRIVAL TIME FOR ALL
SOURCES COMBINED.

X_n = INTERAR. TIME

$Z = \min(X_n)$

Z RV FOR TIME TO NEXT PACKET.

$$1 - F_Z(z) = P[Z \geq z]$$

$$F(z) = P[Z \leq z]$$

$$1 - F(z) = P[\min(X_i) \geq z]$$

$$= P[X_1 > z \; \& \; X_2 > z \; \& \ldots]$$

$$= P[X_1 > z] \, P[X_2 > z] \ldots$$

$$= (1 - F_{X_1}(z))(1 - F_{X_2}(z)) \ldots)$$

SINCE X_i EXPONENTIAL

$$F_{X_i}(z) = 1 - e^{-\lambda_i z}$$

$$1 - F_Z(z) = e^{-\lambda_1 z} e^{-\lambda_2 z} \ldots$$

$$= e^{-z \sum \lambda_i}$$

Z IS EXPONENTIAL w RATE $\sum \lambda_i$

EX 6.11 p 311

USE MAX OF N R.V.

REDUNDANT SYSTEM

N COMPONENTS.

SYSTEM WORKS IF 1 COMPONENT OR MORE IS WORKING.

EACH COMP'S LIFETIME IS EXP. λ_i

X_i : LIFETIME

$P[X_i > k] = e^{-\lambda_i k}$

$F_{X_i}(k) = 1 - e^{-\lambda_i k}$.

W = LIFE OF SYSTEM

$= MAX(X_i)$

$F_W(w) = [F_X(w)]^N = (1 - e^{-\lambda w})^N$

$= 1 - n e^{-\lambda w} + \frac{n^2}{2} e^{-2\lambda w} \cdots$

§ 6-3.1 p318.

MATRICES OF MEANS
COVARIANCES
ETC.

$$X = (X_2)$$

$$m_X = E[X] = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix}$$

MEAN VECTOR

CORRELATION MATRIX

$$R_X = \begin{bmatrix} E[X_1^2] & E[X_1X_2] \\ E[X_1X_2] & E[X_2^2] \\ E[X_1X_3] & \cdots \end{bmatrix}$$

$$(R_X)_{ij} = E[X_i X_j]$$

SECOND ORDER MOMENTS.
BUT CENTRAL MOMENTS MORE USEFUL.

COVARIANCE MATRIX

$$K_X = \begin{bmatrix} E[(X_1-m_1)^2] \\ E[(X_1-m_1)(X_2-m_2)] & \ddots \end{bmatrix}$$

$$(K_X)_{ij} = E[(X_i-m_i)(X_j-m_j)]$$

IF X_i, X_j ARE INDEPENDENT,
THIS TERM IS 0.
(UNCORRELATED IS OK).

6.3.2 LINEAR TRANSFORMATION.
P 320.

EG, PROCESSING COLOR VIDEO.
PIXEL'S COLOR IS VECTOR (R, G, B)
PROCESSING ROTATES (RGB) TO
(Y I Q)

COLOR VECTOR $(R\,G\,B) \to (Y\,I\,Q)$

$$Y = .6R + .3G + .1B$$

$$\underbrace{\begin{pmatrix} Y \\ I \\ Q \end{pmatrix}}_{Y} = A \underbrace{\begin{pmatrix} R \\ G \\ B \end{pmatrix}}_{X}$$

$$Y = AX$$

IF YOU KNOW MEANS FOR X
COVARS
ETC

YOU CAN FIND THEM FOR Y.

$$m_Y = A m_X$$

$$K_Y = A K_X A^T$$

THERE'S A CROSS COVAR
BETWEEN X, Y

$$K_{XY} = E[(X - m_X)(Y - m_Y)]$$

$$Y = AX$$

X_i ARE CORRELATED.
MAYBE WANT A SO Y_i ARE LESS CORRELATED. MAY HELP WITH COMPRESSION.

§6.5 ESTIMATION P332.

E.G. SOURCE X

REC. Y.

WHAT X MAXIMIZES.

$$P[X = x \mid Y = y]$$

i.e. $\max_x P[X = x \mid Y = y]$

MAX A POSTERIORI ESTIMATOR

MAP

$$P[X = x \mid Y = y] = \frac{P[X = x \,\&\, Y = y]}{P[Y = y]}$$

$$= \frac{P[Y = y \mid X = x]\, P[X = x]}{P[Y = y]}$$

BAYES

SOMETIMES WE DON'T KNOW

$P[X=x]$

BUT WE STILL WANT TO DO SOMETHING.

USE MAX LIKELIHOOD EST

ML

$$\max_{x} P[Y=y \mid X=x]$$

(SORT OF ASSUME $f_X(x)$ UNIFORM)