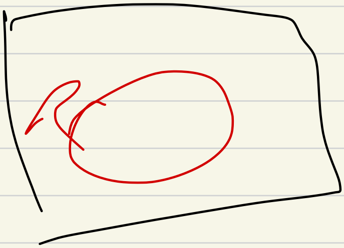
PROB C18 M 3/21/22

EX 5.30 P263 <u>CTD</u>



CHIP WITH DEFECTS

X: RV FOR # DEFECTS

X POISSON $\mu = \alpha$

EACH DEFECT ON CHIP FALLS INTO R <u>w.p.</u> P WITH PROB

Y: RV FOR # DEFECTS IN R.

TELL ME ABOUT $\breve{Y}$.

I.E. WHAT IS $p_Y(i)$?

If $X = K$

$$p_Y(j \mid k) = \begin{cases} 0 & \text{if } j > k \\ \binom{k}{j} p^j (1-p)^{k-j} & 0 \le j \le k \end{cases}$$

$$p_X(k) = \frac{\alpha^k}{k!} e^{-\alpha} \quad \text{Poisson!}$$

$$p_Y(j) = \sum_{\cancel{k=0}}^{\infty} p_Y(j \mid k)\, p_X(k)$$

$k = j$ since $j \le k$

$$= \sum_{k=j}^{\infty} \binom{k}{j} p^j (1-p)^{k-j} \frac{\alpha^k}{k!} e^{-\alpha}$$

$$= \sum_{k=j}^{\infty} \frac{\cancel{k!}}{j!\,(k-j)!} \frac{\alpha^k}{\cancel{k!}} p^j (1-p)^{k-j} e^{-\alpha}$$

$$P(j) = \frac{e^{-\alpha}}{j!} \sum_{k=j}^{\infty} \frac{\alpha^{k}}{(k-j)!} p^{j} (1-p)^{k-j}$$

$$\alpha^{k} = \alpha^{j} \alpha^{k-j}$$

$$= \frac{e^{-\alpha} \alpha^{j} p^{j}}{j!} \underbrace{\sum_{k=j}^{\infty} \frac{[\alpha(1-p)]^{k-j}}{(k-j)!}}_{\text{SLIDE K DOWN BY J}}$$

$$\frac{(\alpha p)^{j} e^{-\alpha}}{j!} \underbrace{\sum_{k=0}^{\infty} \frac{[\alpha(1-p)]^{k}}{k!}}_{e^{+\alpha(1-p)}}$$

$$= \frac{(\alpha p)^{j} e^{-\alpha} e^{\alpha(1-p)}}{j!}$$

$$\cancel{= (\alpha p)^{j}}$$

$$= \frac{(\alpha p)^{j} e^{-\alpha p}}{j!}$$

$$\boxed{\text{POISSON } \mu = \alpha p}$$

EX 5.31 p264

BINARY COMM SYST.

XMIT $X = \begin{cases} +1 & p = \frac{1}{3} \\ -1 & p = 2/3 \end{cases}$

ADD NOISE N GAUSSIAN $\mu = 0$ $\sigma = 1$

REC $Y = X + N$

BIG Q: IF WE SEE SOME Y WHAT DO WE GUESS THAT X WAS?

$P(X = 1 \mid y)$ VS $P(X = -1 \mid y)$.

PICK BIGGER

BAYES

$$F_Y(y \mid X=+1) = P(Y \le y \mid X = 1)$$

$Y = X + N$

$$= P(N+1 \le y \mid \cancel{X=1})$$

$$= P(N \le y-1)$$

N = GAUSSIAN

$$= \int_{-\infty}^{y-1} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x^2}{2}\right)} dx$$

$$\underbrace{P(Y > 0 \mid X = 1)}_{= P(N \ge -1)} = \int_{-1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$= .84$

FROM TABLE

$1 - Q(1)$

Q IS RIGHT TAIL OF GAUSSIAN

$P(Y \geq 0 \mid X = -1)$

$$= \int_{1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = .16$$

$$P(Y > 0) = P(Y > 0 \mid X = 1) P(X = 1) + P(Y > 0 \mid X = -1) P(X = -1)$$

$$= .84 \times \frac{1}{3} + .16 \times \frac{2}{3}$$

$$= .39$$

OBSERVE $P(X = 1) = .33$ THINK

$P(Y > 0) = .39$

$$P(X=1 \mid Y>0)$$

$$= \frac{P(X=1 \;\&\; Y>0)}{P(Y>0)} = P(Y>0 \mid X=1)\, P(X=1)$$

$$P(Y>0) = .39 \qquad P(Y>0 \mid X=1) = .84 \qquad P(X=1) = \frac{1}{3}$$

$$P(X=1 \mid Y>0) = \frac{.84 \times .33}{.39} = .73$$

THIS ISN'T SO BAD BUT

IT'S WRONG 27% OF TIME.

USING Y=0 AS CUTOFF.

IS THERE A BETTER CUTOFF? YES.

CONDITIONAL PDF

EX 5.32 p267

REFERS TO EX 5.8

EX 5.16 p252 ?

$$f(x,y) = 2e^{-x-y} \quad 0 \le y \le x < \infty$$

$$f(x) = \int_x^{\infty} f(x,y)\,dy = 2e^{-x}(1-e^{-x})$$

$$f(y) = \int f(x,y)\,dx = 2e^{-y}$$

CONDITIONAL PDF

$$f_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$f_X(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

$$f(y \mid x) = \frac{2e^{-x-y}}{2e^{-x}(1-e^{-x})}$$

$$= \frac{e^{-y}}{1-e^{-x}} \qquad 0 \le y \le x$$

EX 5.35 p 268

$X = -1 \quad p = 2/3$

$+1 \quad p = 1/3$

$N \sim N(0, 1)$

$Y = X + N$

FOR GENERAL y

WHAT IS $P(X = 1 \mid Y = y)$?

WHAT ABOUT

$$P(X=1 \mid Y=y) = \frac{P(X=1, Y=y)}{P(Y=y)}$$

$$= \frac{P(Y=y \mid X=1)\, P(X=1)}{P(Y=y)}$$

~~BOTH~~ 0

SINCE Y CONTINUOUS.

INSTEAD OF USING $P(Y_i = y)$

WHICH IS 0

USE $P(y < Y_i < y + \Delta)$

TAKE LIMIT AS $\Delta \to 0$

$$P(y < Y_i < y + \Delta) = f_{Y_i}(y)\Delta$$

$$P(X = 1 \mid y < Y_i < y + \Delta)$$

$$= \frac{f(y \mid X = 1)\cancel{\Delta}\left(\frac{1}{3}\right)}{f(y \mid 1)\cancel{\Delta}\frac{1}{3} + f(y \mid -1)\cancel{\Delta}\left(\frac{2}{3}\right)}$$

eg $f(y \mid X = 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}}$ ETC

$$= \frac{1}{1 + 2e^{-2y}} \qquad P(X = 1 \mid Y = y)$$

IF $y = -100$

$$p(x=1 \mid y=-100) = \frac{1}{1+2e^{200}} = 0$$

IF $y = +100$

$$p(x=1 \mid y=+100) = \frac{1}{1+2e^{-200}} = 1$$

IF $y = .35$

THEN $p(x=1 \mid y=.35) = \frac{1}{2}$

USE $y = .35$ AS CUTOFF

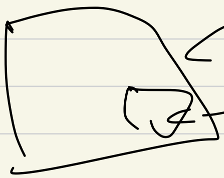
IF $y > .35$ GUESS $x = +1$

$y < .35$ GUESS $x = -1$

5.7.2 CONDITIONAL EXPECTATION p268

$$E[Y|x] = \int_y y\, f_Y(y|x)\, dy$$

EX 5.36 p264



DEFECTS, POISSON α

REGION R p

CHIP

WHAT'S EXPECTED # DEFECTS IN R?

$$E[Y] = \sum_{k=0}^{\infty} \underbrace{E[Y|X=k]}_{kp} P(X=k)$$

$$= \boxed{p \underbrace{\sum_{k=0}^{\infty} k P(X=k)}_{E[X]=\alpha}} = p\alpha$$

$$E[Y] = p \sum_{k=0}^{\infty} k P[X=k]$$

$$= pE[X]$$

$$= p\alpha$$

EX 5.37 $\quad$ p 270

NOISY CHANNEL AGAIN

$E[Y]$?

$$E(Y) = E(Y|X=1)\, P(X=1) + E(Y|X=-1)\, P(X=-1)$$

$$E[Y|X=1] = E[N+1] = E[N]+1 = 1$$

$$P(X=1) = 1/3$$

$$E[Y] = 1 \times \frac{1}{3} + (-1)\frac{2}{3} = -\frac{1}{3}$$

NOTE

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad |x| < \infty$$

TAYLOR SERIES