PROB CLASS 16

M 2022-03-14

REMINDER:

CHAP 5 2 VARIABLES.

GAUSSIAN $\mu = 0$ $\sigma = 1$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$ PROVE IT.

$$\boxed{A = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx}$$

TO SHOW $A = 1$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\,dy$$

$$A^2 = \frac{1}{2\pi} \int e^{-x^2/2} \int e^{-y^2/2}\,dx\,dy.$$

$$A^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2/2} e^{-y^2/2} \, dx\, dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(\frac{x^2+y^2}{2}\right)} \, dx\, dy.$$

CHANGE VARIABLES

$$R = \sqrt{x^2+y^2} \qquad x = R\cos\theta$$
$$y = R\sin\theta$$

$$\frac{dx}{dR} = \cos\theta \qquad \frac{dx}{d\theta} = -R\sin\theta$$

$$\frac{dy}{dR} = \sin\theta \qquad \frac{dy}{d\theta} = R\cos\theta$$

$$\begin{vmatrix} \frac{dx}{dR} & \frac{dx}{d\theta} \\ \frac{dy}{dR} & \frac{dy}{d\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{vmatrix} = R$$

$$A^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(\frac{x^2+y^2}{2}\right)} \, dx \, dy$$

$$= \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \int_{R=0}^{\infty} \frac{1}{R} e^{-\frac{R^2}{2}} \, dR \, d\theta$$

$$= \frac{1}{2\pi \cancel{R}} \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \; \underbrace{\int_0^{\infty} \frac{1}{R} e^{-\frac{R^2}{2}} \, dR}_{1}$$

(HAND WAVING)

MAYBE R BACKWARDS

$$= \frac{2\pi}{2\pi} = 1$$

$$A^2 = 1 \rightarrow A = 1 \qquad \frac{1}{\sqrt{2\pi}} \int e^{\frac{x^2}{2}} \, dx = 1.$$

2 VARIABLE GAUSSIAN.

EXAMPLE 5.18 p253

PARAMETERS: $\mu_x, \sigma_x, \mu_y, \sigma_y, \rho$

ρ: CORRELATION COEFFICENT

$|\rho| \le 1$

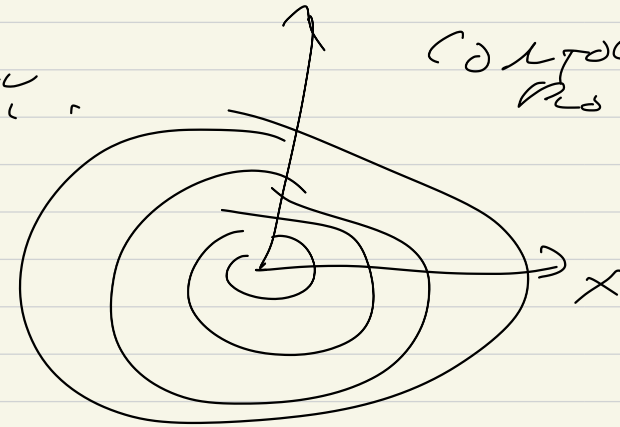
PLOT DENSITY.

CONTOUR PLOT

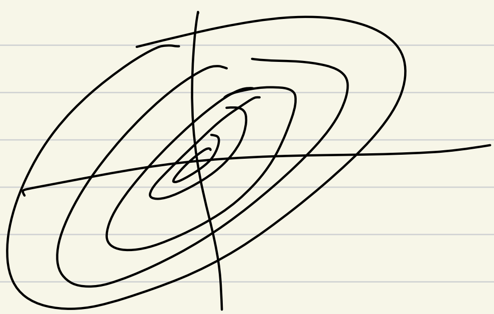
$\mu_x = \mu_y = 0$

$\sigma_x = \sigma_y = 1$

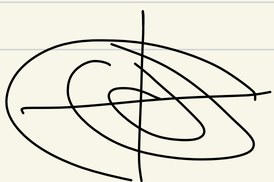
$\rho = 0$



$\rho = \frac{1}{2}$



$\rho = -\frac{1}{2}$



$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\left[\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right]}$$

$$\frac{1}{2\pi\sqrt{1-\rho^2}} \operatorname{EXP}\left(-\left(\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right)\right)$$

(EQ 5.18)

IF $\rho=0$

$$f(x,y) = \frac{1}{2\pi} \operatorname{EXP}\left[-\frac{x^2+y^2}{2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \operatorname{EXP}\left(-\frac{x^2}{2}\right) \frac{1}{\sqrt{2\pi}} \operatorname{EXP}\left[-\frac{y^2}{2}\right]$$

MARGINAL PDF FOR X
FOR GENERAL ρ

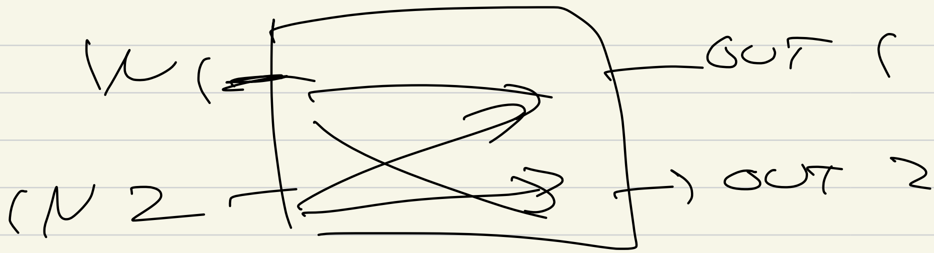
$$f(x) = \int_{y} \frac{1}{2\pi\sqrt{1-\rho^2}} \operatorname{EXP}\left[-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right] dy$$

$$f(x) = \frac{e^{-x^2}}{\sqrt{2\pi}}$$

FOR JOINT GAUSSIAN, MARGINAL PDF IS A SINGLE GAUSSIAN.

EXAMPLE 8.5 p238

PACKET SWITCH



EACH INPUT PORT MAY HAVE A PACKET ARRIVE $p = \frac{1}{2}$ INDEPENDENT.

A PACKET GOES OUT ON EITHER OUTPUT PORT $p = \frac{1}{2}$

OUTCOMES FOR INPUT PORT 1

n = NO INPUT $p = \frac{1}{2}$

$a1$ = INPUT GOING TO OUT 1 $p = \frac{1}{4}$

$a2$ = 2 $p = \frac{1}{4}$

DITTO INPUT PORT 2

X = # PACKETS OUT ON PORT 1
Y = ... 2

Ω = INPUTS.

Ω = (n, n) $(n, a1)$ . . . $(a2, a2)$

(X, Y) = $(0, 0)$ $(1, 0)$ . . . . $(0, 2)$

X = # OUT PACKETS ON PORT 1
Y . . . . PORT 2

$P_{XY}(0,0) = \frac{1}{4}$

$P_{XY}(0,1) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

PACKET IN ON PORT 1 GOING OUT TO PORT 2

NO PACKET IN ON PORT 2

$= \frac{1}{8}$

EX 5.9 P 242

TRANSMIT A MESSAGE N = LENGTH.

MUST SPLIT MESSAGE INTO BLOCKS

OF LENGTH M + PARTIAL LAST BLOCK.

ALSO APPLIES TO FILES ON DISK.

BLOCK SIZE = 4K.

WHOLE BLOCKS $Q = \left\lfloor \frac{N}{M} \right\rfloor$

LENGTH OF FRAGMENT $R = N \bmod M$.

EG. $N = 10000$ $\quad Q = 2$

$M = 4096$ $\quad R = 1808$

N HAS GEOMETRIC DISTN.

$P_N(n) = (1-p)p^n$ $\qquad$ PARAM p

$0 \le n < \infty$

$$\sum_{3}^{\infty} p(n) = 1$$

WHAT'S PROB DIST FOR Q, R?

$$p_N(n) = (1-p)\, p^{n}$$

$$n = qM + r$$

$$P(Q=q, R=r) = (1-p)\, p^{qM+r}$$

$$0 \le q \le \infty \qquad 0 \le r < M.$$

$$P(Q=q) = \sum_{r=0}^{M-1} p(Q=q, R=r)$$

$$= \sum_{r=0}^{M-1} (1-p)\, p^{(qM+r)}$$

$$= (1-p)\, p^{qM} \sum_{r=0}^{M-1} p^{r}$$

$$= \frac{1-p^{M}}{1-p}$$

$$= (1-p^{M})\, p^{qM}$$

GEOMETRIC

DIST OF R?

$$P(Q=q, R=r) = (1-p)\, p^{qM+r}$$

$$P(R=r) = \sum_{q=0}^{\infty} (1-p)\, p^{qM+r}$$

$$= (1-p)\, p^{r} \sum_{q=0}^{\infty} p^{qM} = \frac{1}{1-p^{M}}$$

$$= \frac{1-p}{1-p^{M}}\, p^{r}$$

GEOMETRIC ALSO

$N = qM + r$

$q = \left\lfloor \frac{N}{M} \right\rfloor$ FLOOR.

$r = \mathrm{mod}(N, M)$ MODULO

Ex: $N = 20$
$M = 7$
$q = 2$
$r = 6$

ARE Q, R INDEPENDENT?

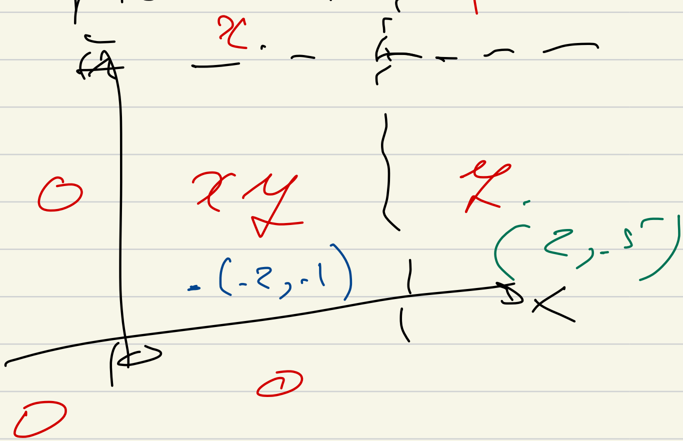
DOES $P(Q=q, R=r) = P(Q=q)\,P(R=r)$?

YES.

JOINT CDF

EX 5.11 PAGE 245

$f_{XY}(x,y) =$



$$F_{XY}(x,y) = P[X \le x,\ Y \le y]$$

$F(.2, .1) = .02$

$F(2, .5) = .5$

DENSITY FOR CONTIN

$$f(x,y) = \frac{d^2}{dx\,dy} F(x,y)$$

IN $0 \le x \le 1$ $F(x,y) = xy$

$0 \le y \le 1$

$f(x,y) = 1$

OUTSIDE, $f(x,y) = 0$

MARGINAL CDF

$$F_X(x) = F_{X,Y}(x,\infty)$$

$$= x \qquad 0 \le x \le 1.$$

$$F_Y(y) = y \qquad 0 \le y \le 1.$$

$$f_X(x) = 1 \qquad 0 \le x \le 1$$

$$f_Y(y) = 1 \qquad 0 \le y \le 1$$

ARE X, Y INDEP?

INTERESTING REGION $0 \le x \le 1$
$0 \le y \le 1$

$f_{XY}(x,y) = 1 = f_X(x)\, f_Y(y)$

$f_X(x) = 1$

$f_Y(y) = 1$

YES.
INDEP.

5.3.1. P247

RANDOM VARS OF DIFFERENT TYPE!

NOISY TRANSMISSION

X: INPUT -1 OR $+1$

ADD NOISE $N \sim U[-2, 2]$

OUT $Y = X + N$

TO GUESS WHAT WAS TRANSMITTED

$P[X=1 \mid Y \le y]$?

$\overline{P[Y \le y \mid X=1]}$

$Y = X + N$

$P[X+N \le y \mid X=1]$

$= P[N \le y-1]$

eg if $y=0$ $P[N \le -1] = \frac{1}{4}$

SINCE $N = U[-2,2]$

ETC. SEE BOOK.

$P[X=1, Y \le 0]$ . . .