PROBABILITY CLASS 11

R 2/17/22

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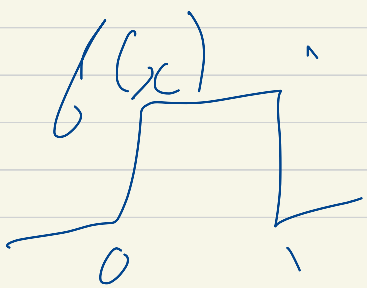
CONTINUING CHAP 4.

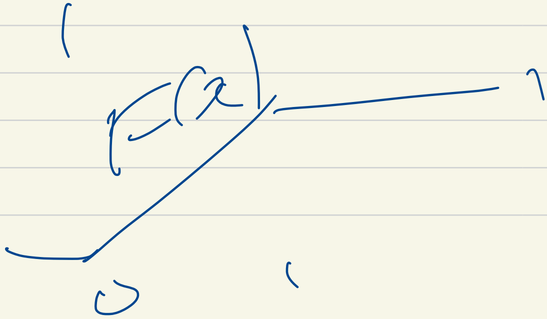
A?? FUNCTION OF R.V.

SAY $X$ IS $U[0,1]$

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & \end{cases}$$



DEFINE $Y = X^2$

WANT $f_Y(y)$, $F_Y(y)$

$$P(Y \le y) = F_Y(y)$$

$$= P(X^2 \le y) = P(X \le \sqrt{y})$$

$$F_Y(y) = F_X(\sqrt{y})$$

IN THIS CASE $F_Y(y) = F_X(\sqrt{y})$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \qquad = \sqrt{y}$$

$$0 \le \sqrt{y} \le 1$$

$$F_Y(y) = F_X(\sqrt{y}) =$$

$$\begin{cases} 0 & \text{if } \sqrt{y} < 0 \\ \sqrt{y} & 0 \to 1 \\ 1 & > 1 \end{cases}$$

$$f_X(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \sqrt{y}$$

$$= \frac{1}{2} y^{-1/2} \qquad 0 \to 1$$

$$\int f_Y(y)\,dy = 1$$

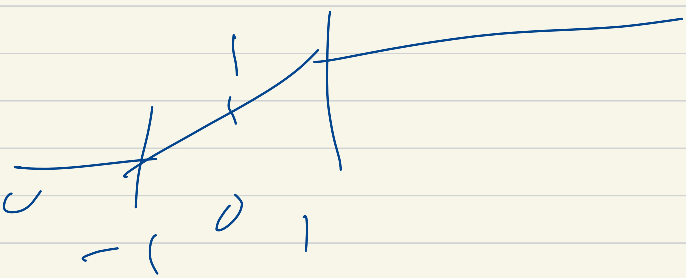
=

$Y = X^2$

But now

$$G_X(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & x \geq 1 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{x+1}{2} \text{ if } & -1 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$



CDF

$F_Y(y) = P(Y \le y) = P(X^2 \le y)$

$= P(|x| \le \sqrt{y})$ $\quad y \ge 0$

2 CASES $x \ge 0$, $x < 0$

$P(|x| \le \sqrt{y}) =$

$P(0 \le x \le \sqrt{y}) + P(-\sqrt{y} \le x \le 0)$

$F_X(\sqrt{y}) - F_X(-\sqrt{y})$

REASONi

$P(-\sqrt{y} \le x \le 0)$

$1 - P(x \le -\sqrt{y}) = 1 - F_X(-\sqrt{y})$

$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy} F_X(\sqrt{y})$$

$$- \frac{d}{dy} F_X(-\sqrt{y})$$

$$= \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}$$

4.6 p181

INEQUALITIES

YOU KNOW SOMETHING ABOUT A R.V., SAY $\mu$, OR $\mu, \sigma$

WHAT CAN YOU CONCLUDE?

$\mu = E[X] = $ MEAN

$\sigma = STD[X] = $ STD DEV

E.G. $S = \{$RPI STUDENTS$\}$

MEASURE HEIGHTS $X$

FIND THAT $E[X] = 5$

CAN WE SAY SOMETHING ABOUT $P[X \geq 10]$?

YES $P[X \geq 10] \leq \frac{1}{2}$

$$\sigma = STD[X] = \sqrt{VAR[X]}$$

$$VAR[X] = E[X^2] - (E[X])^2$$

# MARKOV INEQUALITY

IF $X \geq 0$ $\quad P[X \geq a] \leq \frac{E[X]}{a}$

WITH MORE INFO, YOU COULD TIGHTEN THE BOUND.

$$E[X] = \int_0^\infty x f(x)\,dx$$

$$= \int_0^a x f(x)\,dx + \int_a^\infty x f(x)\,dx$$

$$\geq \int_a^\infty x f(x)\,dx$$

$x \geq a$

$$\geq \int_a^\infty a f(x)\,dx$$

$$= a \left[\int_a^\infty f(x)\,dx = 1 - F(a)\right]$$

$E[X] \geq a P(X \geq a)$

$P(X \geq a) \leq \dfrac{E[X]}{a}$

MARKOV INEQUAL.

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WHAT IF WE ALSO KNOW $\sigma$

NOW $X$ CAN BE NEGATIVE.

CHEBYSHEV INEQUALITY.

$P[|X-\mu| \geq a] \leq \dfrac{\sigma^2}{a^2}$

LET $D = X - \mu$

$P[|X-\mu| \geq a]$

$= P[D^2 \geq a^2]$

$\leq E[D^2] / a^2$

$$P[|X-\mu| \geq a] \leq \frac{E[D^2]}{a^2}$$

$$D = X-\mu$$

$$E[D^2] = \sigma^2$$

$$P[|X-\mu| \geq a] \leq \frac{\sigma^2}{a^2}$$

EX STUDENTS $\mu = 5$

$\sigma = 1$

$\frac{2}{3}$ OF STUDENTS' HEIGHTS

$4 \leq X \leq 6$

(IF GAUSSIAN)

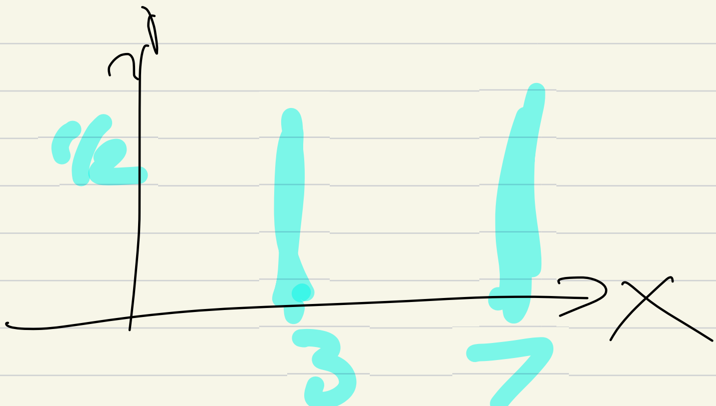
$$P[|H-5| \geq 1] \leq \frac{1}{1} = 1$$

$$P(|X-5| \geq 2) \leq \frac{\sigma^2}{\epsilon^2} = \frac{1}{4}$$

VERY LOOSE BOUND BUT IT ASSUMES NOTHING ABOUT DISTRIBUTION EXCEPT

$\mu = 5 \quad \sigma = 1$

POSSIBILITY




NOT A GOOD EXAMPLE BUT SHOWS IDEA

IF WE KNOW DISTRIBUTION eg GAUSSIAN, WE CAN DO BETTER.

4.7 TRANSFORM METHODS.

p184

SIMILAR TO FOURIER TRANSFORMS.
LAPLACE

a. PROBABILITY GENERATING FN.

ASSUMES $k \geq 0$

DISCRETE

$$G_N(z) = E[z^N]$$

$$= \sum_{k=0}^{\infty} p(k) z^k$$

g. GEOMETRIC R.V.

WITH $p = \frac{1}{2}$

$$p(k) = \frac{1}{2^k} = 2^{-k} \quad k \geq 1$$

$$G(z) = \sum p(k) z^k$$

$$= \sum 2^{-k} z^k$$

$$= \sum \left(\frac{z}{2}\right)^k = \frac{1}{1 - \frac{z}{2}}$$

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$$G(z) = \sum_{k=0} p(k) z^k$$

$$G(0) = p(0)$$

$$\frac{d}{dz} G(z) = \sum_{k=0}^{\infty} \frac{d}{dz} p(k) z^{k}$$

$$= \sum_{k=1}^{\infty} p(k) k z^{k-1}$$

$$\left. \frac{d}{dz} G(z) \right|_{z=0} = p(1)$$

$$p(k) = \left. \frac{d^k}{dz^k} G(z) \frac{1}{k!} \right|_{z=0}$$

$$G(z) = \sum_{k=0}^{\infty} p(k) z^k$$

$$\frac{d}{dz} G(z) = \sum p(k) k z^{k-1}$$

$$\left. \frac{d}{dz} G(k) \right|_{z=1} = \sum_{k=0}^{\infty} k p(k)$$

$$= E[k]$$

HIGHER ORDER MOMENTS ARE SIMILAR eg $E[k^2]$

TRY GEOMETRIC $p = \frac{1}{2}$

$$p(k) = 2^{-k}$$

$$G(z) = \frac{1}{1-\frac{z}{2}} = \left(1-\frac{z}{2}\right)^{-1}$$

$$\frac{dG(z)}{dz} = -\frac{1}{2}(-1)\left(1-\frac{z}{2}\right)^{-2}$$

$$= \frac{1}{2}\left(1-\frac{z}{2}\right)^{-2}$$

IF $Z = 0$ $\quad \frac{1}{2} \quad p(1)$ ——

IF $Z = 1$ $=$ CONTINUE NEXT WEEK.

# 4.8 RELIABILITY p189

DEFINE $R(t)$

= PROBABILITY THAT OBJECT IS STILL ALIVE AT TIME $t$

R.V $T$ = TIME IT DIES.

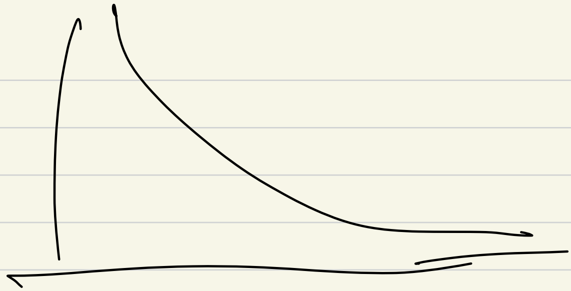
$$R(t) = P(T > t)$$

$$= 1 - F_T(t)$$

$$R'(t) = -f_T(t)$$

SAY EXPONENTIAL

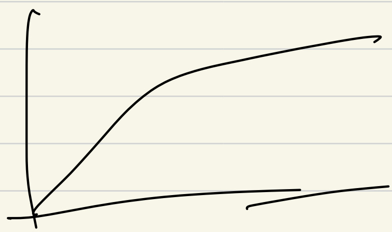
$$f(t) = \lambda e^{-\lambda} \qquad t \geq 0$$

$f(t) =$ 

$$F(t) = \int_0^t \lambda e^{-\lambda t_0}\, dt_0$$

$$= -e^{-\lambda t_0}\Big|_0^t =$$

$$1 - e^{-\lambda t}$$



$$R(t) = 1 - F(t)$$

$=$  $= e^{-\lambda t}$

MTTF

= MEAN TIME TO FAILURE

$= \mathbb{E}[T]$

$= \int R(t)\,dt$

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MAY WANT A CONDITIONAL CDF.

I.E. WHAT'S BULB'S FUTURE GIVEN IT'S STILL ALIVE AT TIME t?

$\therefore F_t(x \mid T > t) =$

$=$

$P(T \le x \mid T \ge t)$

$$F_T(x \mid T > t) = P(T < x \mid T > t)$$

$$\boxed{\text{ASIDE} \quad P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

$$= \frac{P[t \le T \le x]}{1 - F_T(t)} \qquad T \ge t$$

$$= \frac{F(x) - F(t)}{1 - F(t)}$$

PROB BULB IS STILL ALIVE AT T = x GIVEN IT WAS ALIVE AT T = t

EXPONENTIAL

$$F(x) = 1 - e^{-\lambda x}$$

MAKE $\lambda = 1$

$$F(x) = 1 - e^{-x}$$

SAY BULB WAS ALIVE @ $T = 2$.

$$F(x \mid T \geq 2) = \frac{1 - e^{-x} - (1 - e^{-t})}{1 - e^{-t}}$$

$$= \frac{e^{-t} - e^{-x}}{1 - e^{-\cancel{t}}}$$

$$\boxed{\text{(MANGLED SOMETHING)}}$$

$$= \frac{e^{-2} - e^{-x}}{1 - e^{-2}}$$

$= 1.1(-1 - e^{-x})$

$= 1 - 1.1 e^{-x}$