PROBABILITY CLASS 9
R 2/10/22

TEXT CHAP 4

CUMULATIVE DISTRIBUTION FUNCTION CDF

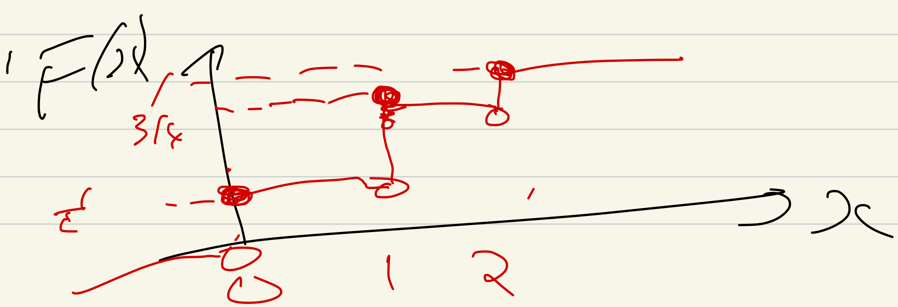
$$F(x) = P\{X \le x\}$$

CAP X ↑ R.V.

x VALUE

E.G. COIN TOSS $N=2 \quad p=\frac{1}{2}$

$$p(0)=\frac{1}{4} \quad p(1)=\frac{1}{2} \quad p(2)=\frac{1}{4}$$



PROPERTIES

$$\lim_{x \to -\infty} F(x) = 0$$

$$\lim_{x \to +\infty} F(x) = 1$$

$$x < y \rightarrow F(x) \leq F(y)$$

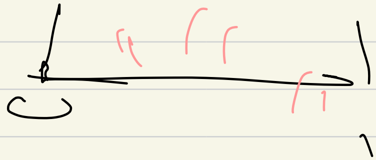
MONOTONE

MOTIVATION FOR CDF:

HANDLES $\left\{ \begin{array}{l} \text{DISCRETE} \\ \text{CONTINUOUS} \\ \text{MIXED} \end{array} \right.$ R.V

CONTINUOUS.

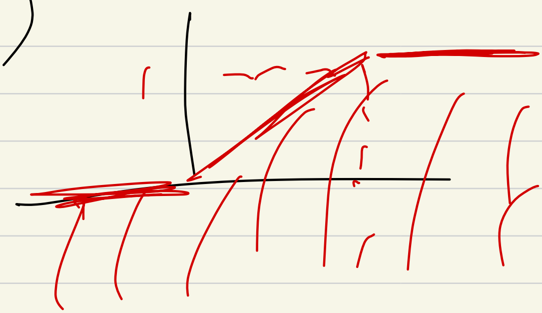
PICK POINT UNIFORMLY ON LINE $0 \le x \le 1$



$$P[X \le x] = \begin{cases} 0 & \text{IF } x < 0 \\ x & \text{IF } 0 \le x \le 1 \\ 1 & \text{IF } x > 1 \end{cases}$$

"

CDF $F(x)$



I WANT A CONTINUOUS ANALOG TO PMF

PROB MASS FN

THAT'S PDF : PROBABILITY
DENSITY
FUNCTION

$f_X(x)$ ← SPECIFIC VALUE.

LITTLE f

NAME OF R.V.

FOR UNIFORM $F(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$

PDF $f(x) = \dfrac{dF(x)}{dx}$ IF IT EXISTS.

IF $F(x)$ JUMPS, $f(x)$
DOESN'T EXIST THERE.

$F = F(x)$

$$\frac{dx}{dx} = 1$$

PDF:

$f(x) =$

$$P(x_0 \le X \le x_0 + dx)$$

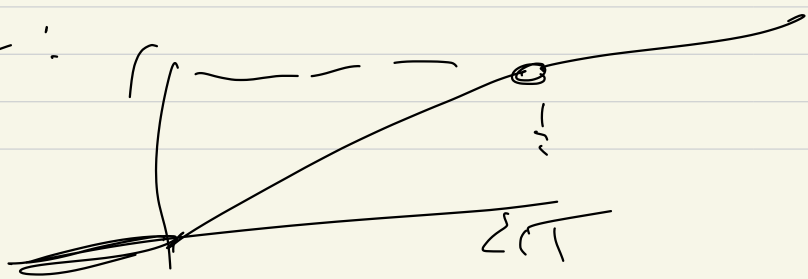
$$= f(x_0)\,dx$$

EXPERMENT: ROTATE A POINTER

MEASURE ANGLE θ

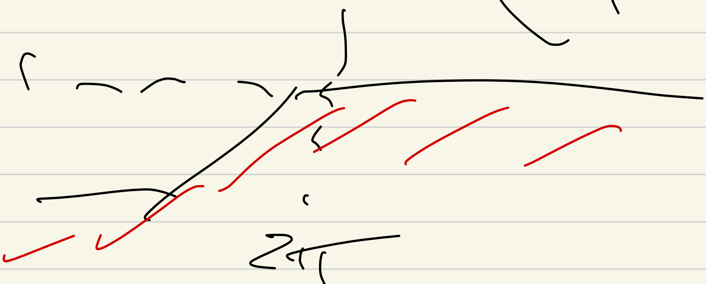
$$0 \le \theta \le 2\pi$$

CUMUL:

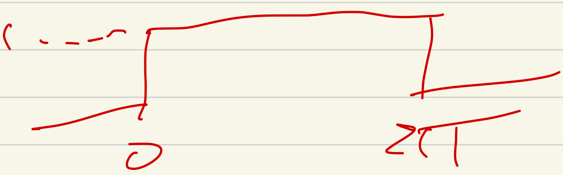


CDF

$$F(\theta) = \begin{cases} 0 & \text{if } \theta \le 0 \\ \dfrac{\theta}{2\pi} & \text{if } 0 \le \theta \le 2\pi \\ 1 & \text{if } \theta \ge 2\pi \end{cases}$$



$$f(\theta) = \frac{d}{d\theta} F(\theta) = \begin{cases} 0 & \theta \le 0 \\ \dfrac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & 1 \le \theta \end{cases}$$



$$F(x) = \int_{-\infty}^{x} f(x_1)\,dx_1$$

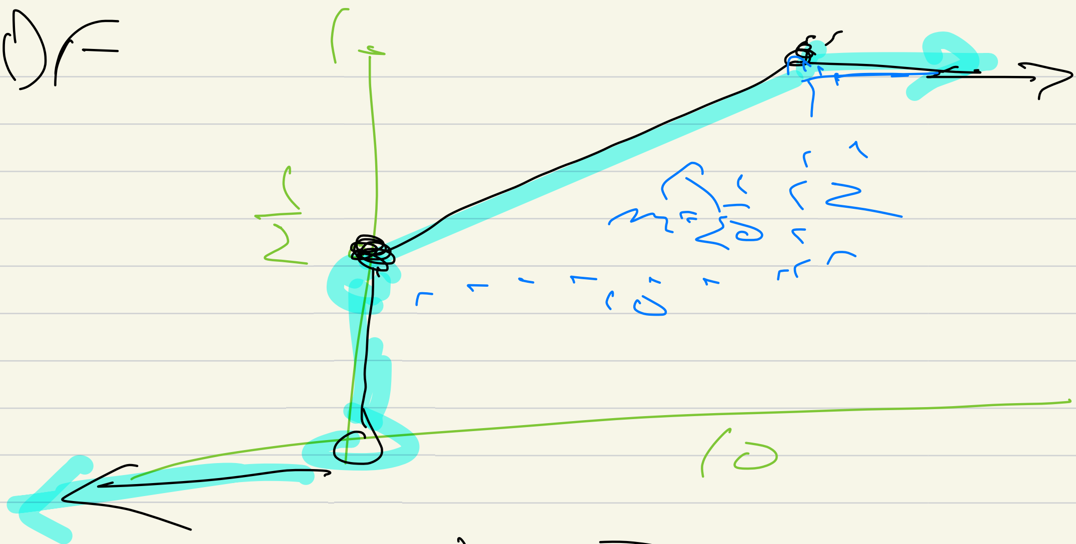
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

MIXED RANDOM VARIABLES

- WANT AN UBER.
  50% IT'S HERE NOW
  50% I WAIT FROM 0 TO 10 MINUTES.
  UNIFORMLY

RANDOM VAR: WAIT TIME

CDF



$$P(x < 0) = 0$$

$$P(X = 0) = \frac{1}{2}$$

$$P(0 \le X \le 10) = \frac{1}{2} + \frac{X}{20}$$

$$P(X > 10) = 1$$

$$\boxed{F(x)}$$

DENSITY FUNCTION

$$\frac{dF(x)}{dx} = f(x)$$

$$f(x) = \begin{cases} 0 & \text{IF } x < 0 \\ \text{UNDEFINED} & \text{IF } x = 0 \\ \frac{1}{20} & \text{IF } 0 < x < 1 \\ \text{UNDEF} & \text{IF } x = 1 \\ 0 & \text{IF } x > 1 \end{cases}$$

Q: WHAT'S PROBABILITY I WAIT FROM 8 TO 7 MINUTES?

A1 : $F(7) - F(5) = \frac{2}{20} = \frac{1}{10}$

$\cancel{5}$ $\frac{7}{2} + \frac{7}{20} = \frac{17}{20}$ $\quad \frac{5}{2} + \frac{5}{20}$

$= \frac{15}{20}$

A2 $\int_5^7 f(x)\,dx = \int_5^7 \frac{1}{20}\,dx$

$= \frac{2}{20} = \frac{1}{10}$

EXPONENTIAL R.V.

$$x \geq 0$$

$$P[X > a] = e^{-\lambda a}$$

CHECK $P[x \geq 0] = e^{-0} = 1$ ✓

$$P[X \geq \infty] = e^{-\lambda \infty} = e^{-\infty} = 0$$ ✓

$$F[x] = P[X \leq x] = 1 - e^{-\lambda x}$$

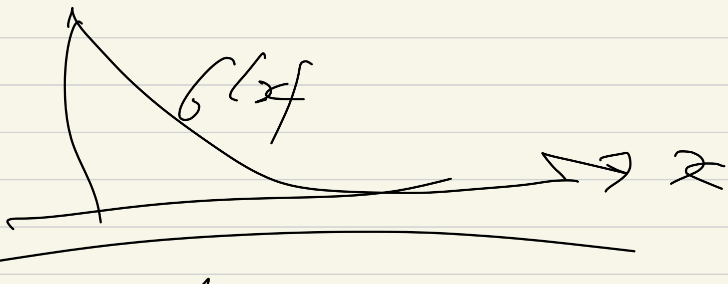
EXPONENTIAL R.V. IS CONTINUOUS VERSION OF DISCRETE GEOMETRIC R.V.

E.G. PROBABILITY THAT THIS ^{238}U ATOM HAS NOT DECAYED BY TIME x.

$$F(x) = 1 - e^{-\lambda x} \qquad x \geq 0$$

PDF $f(x) = \frac{d(F(x))}{dx}$

$$= -(-\lambda) e^{-\lambda x} = \lambda e^{-\lambda}$$



ANOTHER APP:

INTERARRIVAL TIMES.

R.V TIME BETWEEN CALLS TO CALL CENTER OR 2 ATOMS DECAYING ETC.

COMPLEMENTARY TO POISSON

IF TIME TO PROCESS A CALL IS GREATER THAN EXPECTED TIME BETWEEN CALLS, THEN YOU HAVE A PROBLEM.

$$E[X] = \int^{\infty} x \lambda e^{\lambda x} \, dx$$

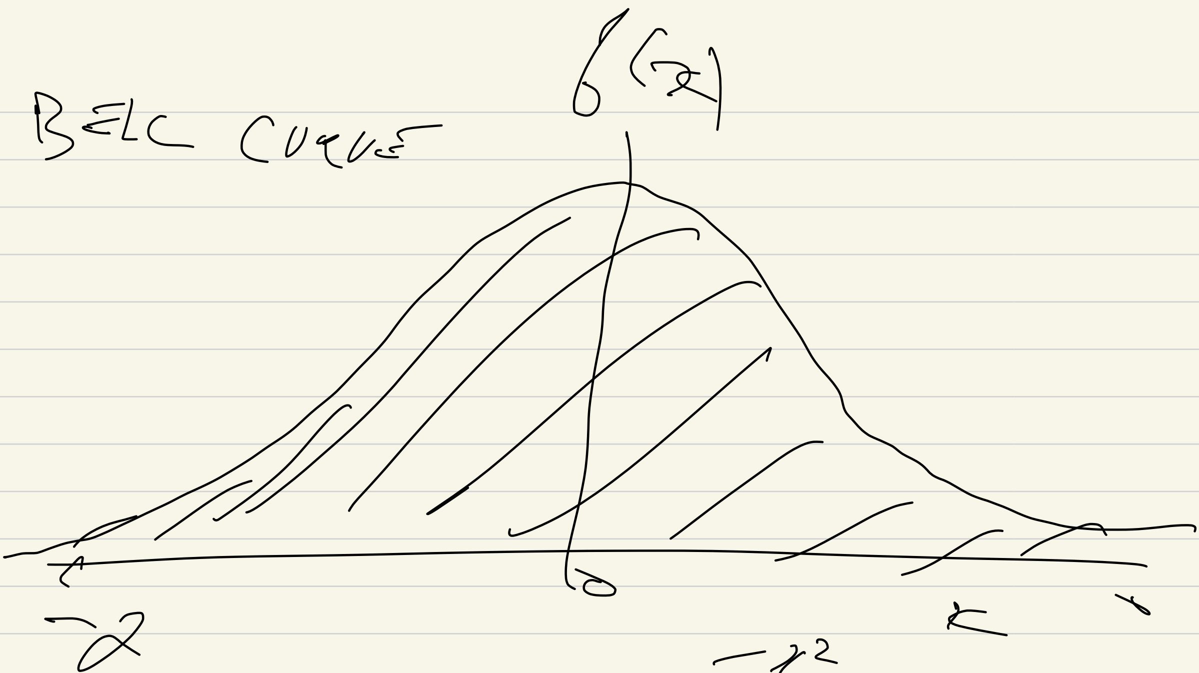
THE MOST IMPORTANT CONTINUOUS R.V. GAUSSIAN

AKA NORMAL

AS $N \to \infty$, ALMOST EVERY OTHER DISTRIBUTION STARTS LOOKING LIKE THE GAUSSIAN

"LAW OF LARGE NUMBERS"

BELL CURVE



$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$\mu = 0 \quad \sigma = 1$

CDF $F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_0^2}{2}} \, dx_0$

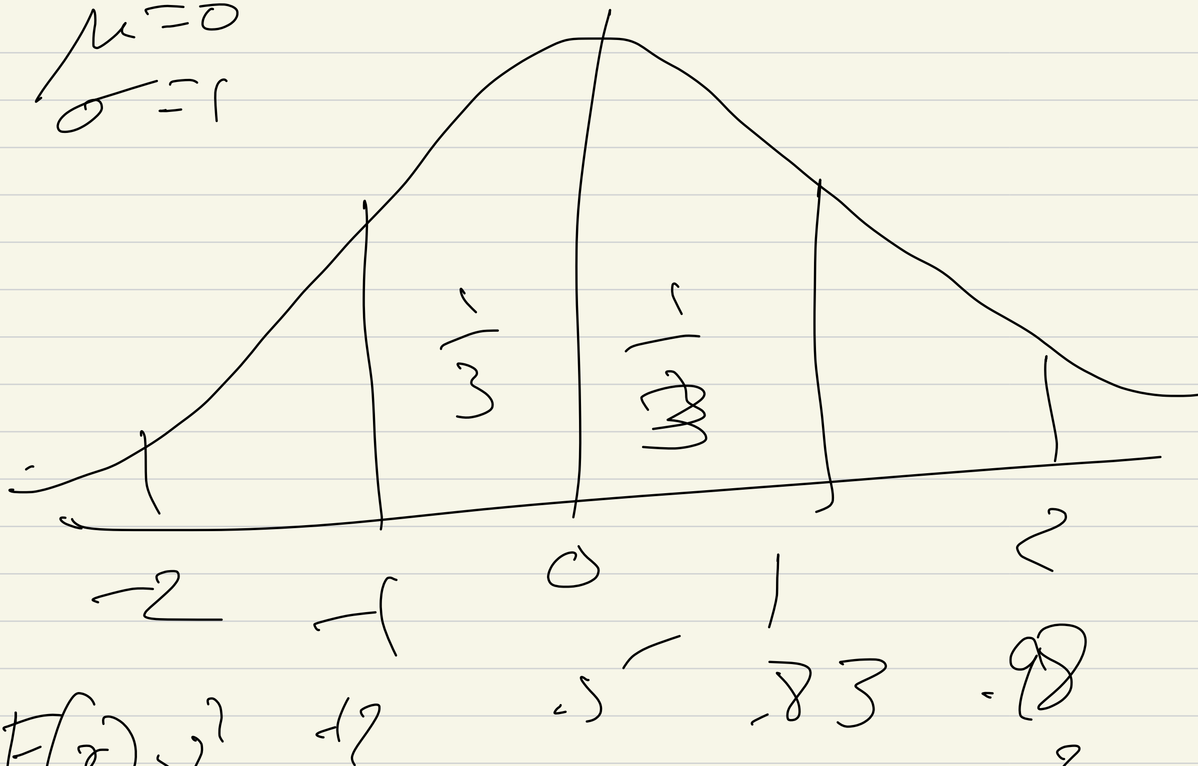
NO CLOSED FORM. USE CALCULATOR OR TABLES.

GENERAL μ, σ

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left[\frac{(x-\mu)^2}{2\sigma^2}\right]}$$

SAT $\mu = 500$ $\sigma = 100$

$\mu = 0$
$\sigma = 1$



$F(x), \delta^2$,

TO APPROX BINOMIAL,

MATCH μ, σ

$\mu = np$ $\quad p = \frac{1}{2}$ $\mu = \frac{n}{2}$

$\sigma = \sqrt{npq}$ $\quad \sigma = \frac{\sqrt{n}}{2}$

IT'S GOOD FOR SMALL N

ALREADY SAY ALMOST

TO APPROX POISSON, WANT

α BIG ENOUGH THAT THERE'S

A LEFT TAIL

POISSON $\mu = \alpha$

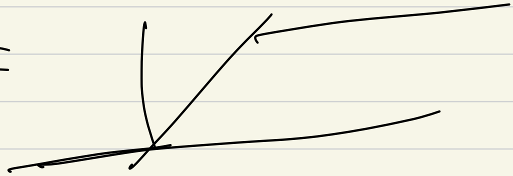
$\sigma = \sqrt{\alpha}$

APPLICATION OF CDF.

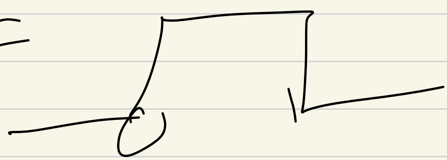
EXPT 1 PICK 2 UNIFORM R.V.
IN INTERVAL [0,1]
REPORT BIGGER
R.V. IS SMALLER ONE.
WHAT'S ITS DISTRIBUTION.

LET U = 1ST R.V.
V = 2nd.
$W = MAX(U,V)$.

U: CDF



V: PDF



$X = \max(U, V)$

$X \le x$ IFF $U \le x$
AND $V \le x$

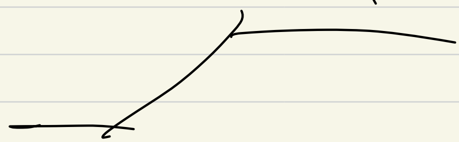
PROB $X \le x$ =
PROB $U \le x$ × PROB $V \le x$

BOTH $U \le x$ AND $V \le x$
PROBABILITY IS PRODUCT
OF PROBS (U, V INDEPENDENT)

$$F_X(x) = F_U(x)\,F_V(x)$$

$$F_U(x) = x \qquad 0 \le x \le 1$$

$$F_U(x) = x$$



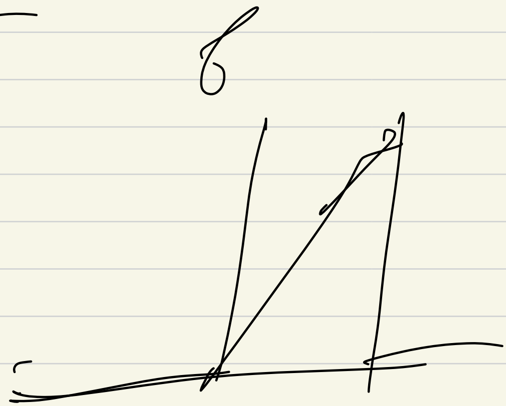
$$F_X(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ 0, & x \le 0 \\ 1, & x \ge 1 \end{cases}$$

MAX OF K UNIFORM R.V.

CDF $F_K(x) = x^k \qquad 0 \le x \le 1$

$N = 2 \quad F(x) = x^2$

DENSITY $f(x) = 2x$



BIASED TO HIGH.

MAX OF 2 UNIFORM R.V.

MORE LIKELY TO BE HI.

THAN LO

X = MIN OF 2 UNIFORM R.V.

$$1 - F_2(x) = [1 - F_1(x)]^2$$

$$(1-x)^2$$

$$F_2(x) = 1 - (1-x)^2$$

$$= 2x - x^2$$

$$f(x) = 2 - 2x$$