M2 1/7/22

PROBABILITY CLASS 8

WORK OUT SOME VARIANCES

BINOMIAL $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

$$E(k) = \sum k\, p(k)$$

$$= \sum k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \underbrace{\sum \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}}_{=1}$$

$= np$

I'M SKIPPING OVER THE END CASES OF $K=0$ ETC. HOWEVER THEY DO WORK OUT.

$$VAR[K] = E[K^2] - (E[K])^2$$

$$K^2 = K(K-1) + K$$

$$E[K^2] = E[K(K-1)] + E[K]$$

$$= \sum K(K-1)\binom{n}{K} p^K (1-p)^{n-K}$$

$$= n(n-1)p^2 \sum \frac{(n-2)!\, p^{K-2}}{(K-2)!\,(n-K)!} (1-p)^{n-K}$$

IN MORE DETAIL:

$E[k(k-1)]$ L3

$$= \sum k(k-1)\binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum \frac{k(k-1)\, n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

$$\frac{1}{(k-2)!}$$

$$= p^2 p^{k-2}$$

$$= \sum \frac{n!}{(k-2)!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= n(n-1) \sum \frac{(n-2)!}{\;}$$

$$= n(n-1)p^2 \sum \frac{(n-2)!}{(k-2)!(n-k)!} p^{k-2}(1-p)^{n-k}$$

$$n(n-1)p^2 \underbrace{\sum \binom{n-2}{k-2} p^{k-2}(1-p)^{n-k}}_{=1}$$

$$E[k(k-1)] = n(n-1)p^2$$

$$E[k^2] = E[k(k-1)] + E[k]$$

$$= n(n-1)p^2 + np$$

$$\mathrm{VAR}[K] = E[K^2] - (E[K])^2$$

$$= n(n-1)p^2 + np - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= n(p - p^2) = np(1-p)$$

$$= npq \qquad \text{IF } q = 1-p.$$

ABBREV

$$\mu = E[K]$$

$$\sigma^2 = \mathrm{VAR}[K]$$

$$\sigma = \mathrm{STD}[K]$$

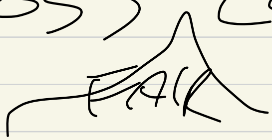
$$= \sqrt{\mathrm{VAR}[K]}$$

ROUGHLY

$$P[\mu - \sigma \le k \le \mu + \sigma] = \frac{2}{3}$$

$$P[k \ge \mu + \sigma] = \frac{1}{6}$$

TOSS COIN 100 TIMES



$n = 100$ $\qquad \mu = np = 50$

$p = \frac{1}{2}$ $\qquad \sigma^2 = npq\ 25$

$= q$ $\qquad \sigma = 5$

$\frac{2}{3}$ OF TIME HEADS

IS 50 ± 5

$50 \pm 10\%$

TOSS COIN 25 TIMES

$n = 25$ $\quad \mu = 12\frac{1}{2}$

$p = \frac{1}{2}$ $\quad \sigma^2 = \frac{25}{4}$

$$\sigma = \frac{5}{2}$$

2/3 OF TIME ~~#~~ HEADS IS

$$12\frac{1}{2} \pm 2\frac{1}{2}$$

$$\pm 20\%$$

TOSS 10000 TIMES

$n = 10000$ $\quad \mu = 5000$

$p = \frac{1}{2}$ $\quad \sigma = 50$

$$5000 \pm 50$$

$$\pm 1\%$$

GEOMETRIC

R.V. IS TOSS # THAT GIVES
1ST HEAD

p = PROB THAT ONE TOSS
SHOWS HEAD

$$p(k) = p(1-p)^{k-1} = pq^{k-1}$$

$$k \geq 1$$

COUNTABLY INFINITE

$$E[k] = \sum_{k=1}^{\infty} k\,p(k) = p \sum k(1-p)^{k-1}$$

?

WANT $\sum k q^{k-1}$

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q} = (1-q)^{-1}$$

DERIVATIVES $\frac{d}{dq} \sum q^k = \sum k q^{k-1}$

$$\frac{d}{dq}(1-q)^{-1} = (1-q)^{-2}$$

$$\sum k q^{k-1} = (1-q)^{-2}$$

$$E[k] = \sum k p q^{k-1} = p(1-q)^{-2} = \frac{1}{p}$$

FAIR $p = \frac{1}{2}$

$p[k=1] = \frac{1}{2}$

$p(k=2) = \frac{1}{4}$

$p(k=3] = \frac{1}{8}$

$E[k] = 2$

$$VAR[K] = E[K^2] - (E[K])^2$$

$$K^2 = K(K-1) + K$$

$$\sum K q^{K-1} = (1-q)^{-2}$$

DERIVATIVES AGAIN

$$\frac{d}{dK} : \sum K(K-1) q^{K-2} = 2(1-q)^{-3}$$

$$\sum K(K-1) q^{K} = 2q^2(1-q)^{-3}$$

I NEED p

$$\sum K(K-1) p^{K} = 2p^2(1-p)^{-3}$$

$$= E[K(K-1)] = 2p^2 q^{-3}$$

$$VAR[K] = E[K(K-1)] + E[K] - (E[K])^2$$

$$= 2p^2q^{-3} + \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1}{p^2}\left[2\frac{p^4}{q^3} + p - 1\right]$$

$$= \frac{1}{p^2}\left[2\,\frac{(1-q)^4}{q^3} - q\right]$$

$$= \frac{1}{p^2}\left[\frac{2}{q^3} - \frac{8}{q^2} + \frac{12}{q} - 8 + 2q - q\right]$$

CONTINUE ON THURS.

CONDITIONAL

FACTORY MAKES HI QUALITY AND LO-Q PARTS.

$\alpha = P[\text{HI Q}]$

$1-\alpha = P[\text{LO Q}]$

BOTH TYPES LIFETIME IS GEOMETRIC.

HI: $P[k] = r(1-r)^{k-1}$ $\quad E[k] = \frac{1}{r}$

$P[k|\text{HI}]$

LO: $P[k] = s(1-s)^{k-1}$ $\quad E[k] = \frac{1}{s}$.

$= P[k|\text{LO}]$

WHAT'S $E[K]$ FOR UNKNOWN PART?

$$E[K] = E[K \mid HI]\, p[HI] + E[K \mid LO]\, p[LO]$$

$$= \frac{1}{r}\alpha + \frac{1}{s}(1-\alpha)$$

$$VAR[K] = E[K^2] - (E[K])^2$$

$$E[K^2] = E[K^2 \mid HI]\,\alpha + E[K^2 \mid LO]\,(1-\alpha)$$

YOU HAVE A PART THAT'S STILL ALIVE AT TIME $\bar{S}$.

WHAT'S PROB IT'S A HQ PART?

$$p(K) = p(K \mid HI)\, p(HI) + p(K \mid LO)\, p(LO)$$

$$= r(1-r)^{k-1}\alpha + s(1-s)^{k-1}(1-\alpha)$$

$$p(K \,\&\, HI) = p(K \mid HI)\, p(HI)$$

$$= r(1-r)^{k-1}\alpha$$

$$p(k \,\&\, H1) = p(H1 \,\&\, k)$$

$$= p(H1 \mid k)\, p(k)$$

$$p(H1 \mid k) = \frac{p(k \,\&\, H1)}{p(k)}$$

$$= \frac{r(1-r)^{k-1}\alpha}{r(1-r)^{k-1}\alpha + s(1-s)^{k-1}(1-\alpha)}$$

PROBABLITY THAT IT'S A HI-Q PART IF IT DIES AT TIME K.

AS K GETS BIGGER THIS GETS LARGER $\rightarrow 1$.

POISSON EXAMPLE

$$p(k) = \frac{\alpha^k}{k!} e^{-\alpha}$$

$E[k] = \alpha$

$p(k)$ = PROB OF k EVENTS HAPPENING IN 1 MINUTE

WHERE α = AVERAGE # IN 1 MINUTE.

CALLS TO A CALL CENTER.

$\alpha = 10$ /MINUTE

ASSUMES THAT POTENTIAL CALLERS ARE INDEPENDENT

MAYBE $N = 300{,}000{,}000$ POSSIBLE CALLERS.

p OF ANY SPECIFIC ONE CALLING IN NEXT MINUTE IS $p = \frac{1}{30{,}000{,}000}$

EXPECTED # $Np = 10$.

PROB OF ~~EX~~ EXACTLY 1 CALL THIS MINUTE

$$\frac{\alpha^1}{1!} e^{-\alpha} = 10 e^{-10}$$

~~GOFA~~ P[10 CALLS IN MINUTE] $\dfrac{10^{10}}{10!} e^{-10}$

I WANT PROBABILITIES FOR AN HOUR-LONG INTERVAL -

MEAN FOR HOUR = 60 MEAN FOR MINUTE

$\beta = 60\alpha$

$\alpha = 10 \Rightarrow \beta = 600$

P[1000 CALLS IN HOUR]
k

$$\frac{\beta^k}{k!} e^{-\beta} = \frac{600^{1000}}{1000!} e^{-600}$$

NEW IDEAS IN CHAP 4.

CDF CUMULATIVE DISTRIBUTION FUNCTION.

NOTATION IS UPPERCASE LETTERS

LIKE $F[k]$ $G(k)$ $G(x)$ ...

FOR R.V. K

$$F[k] = P[K \le k]$$

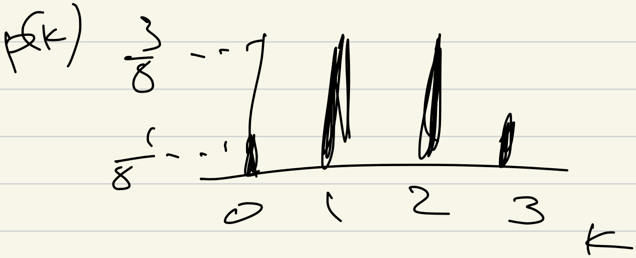
LITTLE k

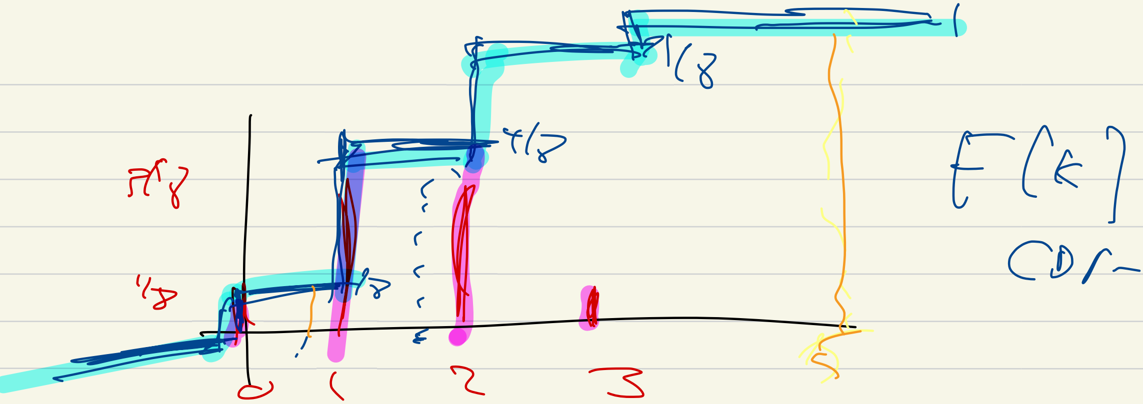
BIG K

$$F[x] = P[X \le x]$$

TOSS FAIR COIN THRICE

$p(0) = \frac{1}{8}$ $p(1) = \frac{3}{8}$, $p(2) = \frac{3}{8}$, $p(3) = \frac{1}{8}$





ASK WHAT'S PROB # HEADS ≤ 1.8

$F[1.8] = \frac{1}{2}$

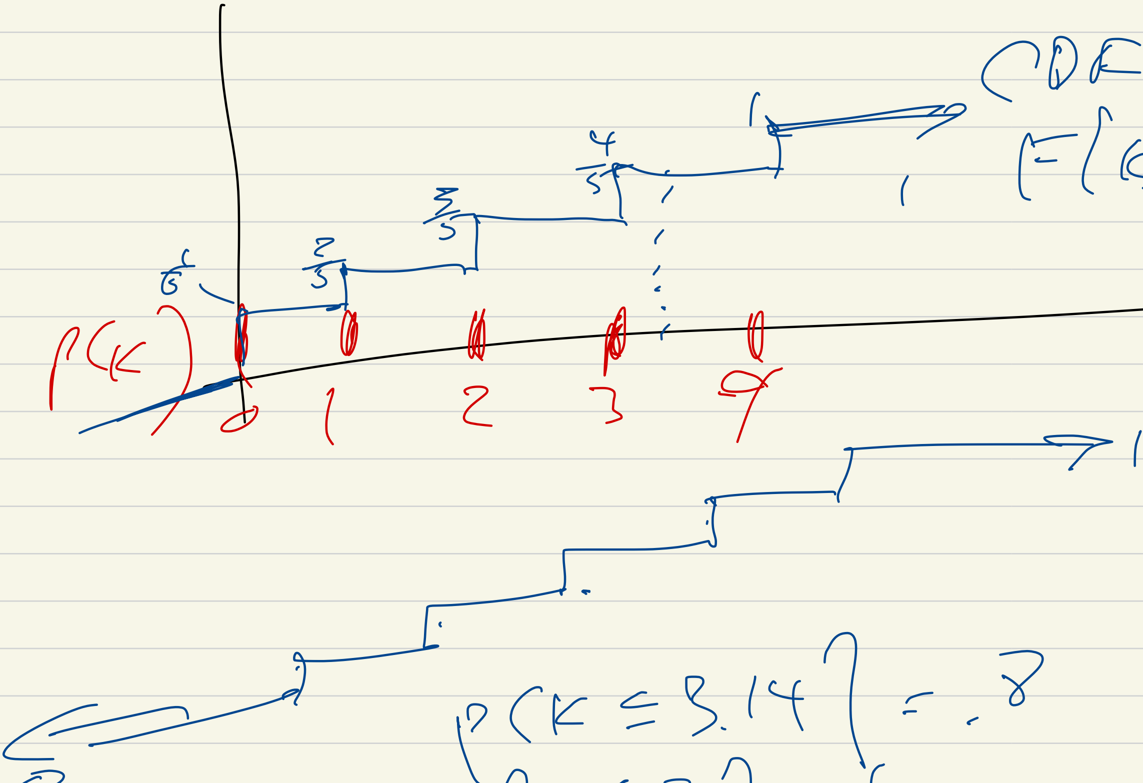
WHATS PROB # HEADS BETWEEN 2/3 AND 5

$F[5] - F[\frac{2}{3}] = 1 - \frac{1}{8} = 7/8$

MONOTONE

UNIFORM R.V. $[0, 4]$

$p(0) = p(1) \cdots p(4) = \frac{1}{5}$



$P(K \le 3.14) = .8$

$P(K \le 20) = 1$

$P(K \le -5) = 0$

$P[3.14 < K \le 20] = 1 - .8 = .2$

CUMULATIVE DISTRIBUTION FUNCTION