PROB 2/3/22 - 1

POISSON DISTRIBUTION

eg. RADIOACTIVE DECAY IN A BLOCK OF RADIUM.

THERE ARE MANY ATOMS, EACH VERY UNLIKELY TO DECAY IN NEXT SECOND.

BUT SOME WILL.

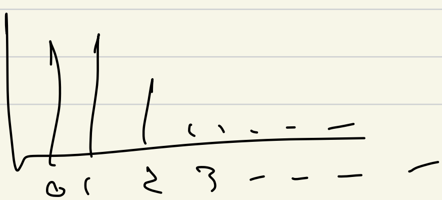
α = EXPECTED NUMBER TO DECAY IN NEXT SECOND

$$P(K \text{ WILL DECAY}) = \frac{\alpha^k}{k!} e^{-\alpha}$$

$$\alpha = 1. \qquad p(k) = \frac{1^k}{k!} e^{-1} \approx \frac{.3}{k!}$$

$p(0) = .3$ $\quad$ $p(2) = .15$

$p(1) = .3$ $\quad$ $p(3) = .05$ ...



$\alpha = 2$

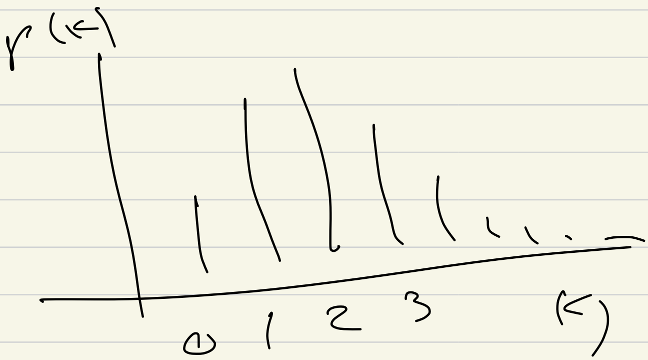
$$p(k) = \frac{2^k}{k!} e^{-2} \approx .1 \frac{2^k}{k!}$$

$p(0) = .1$

$p(1) = .2$

$p(2) = .2$

$p(3) = 1.25$



$$p(k) = \frac{\alpha^k}{k!} e^{-k}$$

CHECK $\sum_{k=0}^{\infty} p(k) = \sum \frac{\alpha^k}{k!} e^{-k} \overset{?}{=} 1$

$$\boxed{\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} = e^{\alpha}}$$

$$E[k] = \sum_{k=\cancel{0}\;1}^{\infty} k\,p(k) = \sum k \frac{\alpha^k}{k!} e^{-\alpha}$$

$$= \alpha e^{-\alpha} \sum_{k=\cancel{0}}^{\infty} \frac{1}{(k-1)!} \alpha^{k-1}$$

I CAN CHANGE LOWER LIMIT FROM 0 TO 1 BECAUSE $k\,p(k) = 0$ WHEN $k=0$

$$\sum_{k=0}^{\infty} k\,p(k) = \sum_{k=1}^{\infty} k\,p(k)$$

$$E[k] = \alpha e^{-\alpha} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \alpha^{k-1}$$

$$= \alpha e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} = e^{\alpha}$$

REPLACE $k-1$ BY k EVERYWHERE.

$$E[k] = \alpha e^{-\alpha} e^{\alpha} = \alpha$$

FOR VAR[K] FOR POISSON

NEED $E[K^2]$

$$= \sum_{K=1}^{\infty} K^2 \frac{\alpha^K}{K!} e^{-\alpha}$$

$$= e^{-\alpha} \left[\sum \left(K + K(K-1) \right) \frac{\alpha^K}{K!} \right]$$

$$= e^{-\alpha} \left[\sum \frac{K\alpha^K}{K!} + \sum \frac{K(K-1)\alpha^K}{K!} \right]$$

$$= e^{-\alpha} \left[\alpha \sum \frac{\alpha^{K-1}}{(K-1)!} + \alpha^2 \sum \frac{\alpha^{K-2}}{(K-2)!} \right]$$

$$= e^{-\alpha} \left[\alpha e^{\alpha} + \alpha^2 e^{\alpha} \right]$$

$$= \alpha + \alpha^2$$

MUST CHECK BOUNDARIES

$$E[K^2] = \alpha + \alpha^2$$

$$VAR[K] = \underbrace{E[K^2]}_{\alpha+\alpha^2} - \underbrace{E[K]^2}_{\alpha^2}$$

$$= \alpha$$

FOR POISSON

$$E[K] = VAR(K) = \alpha$$

$$STD[K] = \sqrt{\alpha}$$

FOR BIGGER α, R.V. IS MORE TIGHTLY CLUSTERED, AS A PERCENTAGE.

IF $\alpha = 1000$ $\quad\underbrace{970}_{E-S} - \underbrace{1030}_{E+S}$

ABOUT 2/3 OF TIME

COMMUNICATION EXAMPLE.

$N = 48$ CUSTOMERS FOR A SHARED CHANNEL.

$\alpha = \frac{1}{3}$ OF THEM TALK AT ANY GIVEN TIME.

Q: HOW MANY CHANNELS SHOULD COMPANY PROVIDE?

IF CUSTOMERS ARE INDEPENDENT

LET R.V. K = # TALKING NOW

$$p(K) = \binom{48}{K} \alpha^K (1-\alpha)^{48-K}$$

$\boxed{\text{BINOMIAL}}$

MEAN?

$$E[K] = \sum_{K=0}^{48} K \alpha^K (1-\alpha)^{48-K}$$

$$= 48\alpha = \underline{16}$$

~~ASIDE~~

BINOMIAL IS EXACT

IF N IS VERY LARGE AND α VERY SMALL AND K SMALL

THEN $\boxed{P(K) = \binom{N}{K} \alpha^K (1-\alpha)^{N-K}}$

IS HARD TO COMPUTE

IF $E(K)$ IS SMALL NUMBER THEN POISSON IS A GOOD APPROX.

(HERE α IS DIFFERENT)

ANOTHER APPROX IS GAUSSIAN (NORMAL) DIST.

IF $N = 6 \times 10^{23}$

$K = 5$

$\binom{6 \times 10^{23}}{5}$ IS A MESS

APPROX FOR $N! = \dfrac{N^N}{e^N} \sqrt{2\pi N}$

STIRLING APPROX

TRY $N=2$ $\qquad \frac{2^2}{e^2}\sqrt{4\pi} \qquad \frac{2\sqrt{4\pi}}{e^2}$

EXACT IS 2. $\qquad$ 1.919

RELATIVE ERROR $\frac{1}{12N}$ $\qquad$ NOT BAD

BACK TO COMMUNICATION EX.

48 CUSTOMERS TALK $\frac{1}{3}$ OF TIME

M CHANNELS.

HOW MANY UNHAPPY CUSTOMERS?

IF $M=48$ NONE.

R.V $K = \#$ TALKING CUSTOMERS.

IF $K \le M$ ALL HAPPY.

IF $K > M$ $\qquad K-M$ UNHAPPY.

LET $U = \#$ UNHAPPY.

$$U = (K - M)^+ \leftarrow \text{CLIPPED AT } 0$$

$$x^+ = \begin{cases} x & \text{IF } x \geq 0 \\ 0 & \text{IF } x < 0 \end{cases}$$

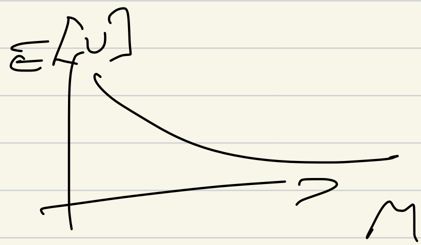
WANT $E[U]$

$$\sum_{i=0}^{48} (K - M)^+ \binom{48}{K} \alpha^K (1-\alpha)^{48-K}$$

$$E[U] = \sum_{K=M+1}^{48} (K - M) \binom{48}{K} \alpha^K (1-\alpha)^{48-K}$$

NO SIMPLE FORM.

YOU CAN PLOT



CONDITIONAL PMF

PMF = PROBABILITY MASS FN.

$$P(A|B) = \frac{P(AB)}{P(B)}$$

X IS R.V. UNIFORM IN $[0, N-1]$

$$P(K) = \begin{cases} \frac{1}{K} & 0 \le K \le N-1 \\ 0 & \text{OTHERWISE} \end{cases}$$

K IS AN INTEGER.

SAY $N = 10$ $\quad P(K) = \frac{1}{10}$

SAY WE KNOW THAT $X \ge 5$

WHAT IS $P(X \mid X \ge 5)$?

$$\frac{P(X \cap X \ge 5)}{P(X \ge 5)} \qquad 0\ 1\ 2\ 3\ 4\ |\underline{5\ 6\ 7\ 8\ 9}$$

IF $X < 5$ $\quad 0$

IF $X \ge 5$ $\quad \rightarrow \frac{\frac{1}{10}}{\frac{1}{2}} = \frac{1}{5}$

UNIFORM $[0 \ldots N-1]$

N CHOICES.

K IS A POSSIBLE RESULT

$$P(K) = \begin{cases} \frac{1}{N} & \text{IF } 0 \le K < N \\ 0 & \text{OTHERWISE} \end{cases}$$

TOSS COIN 3 TIMES

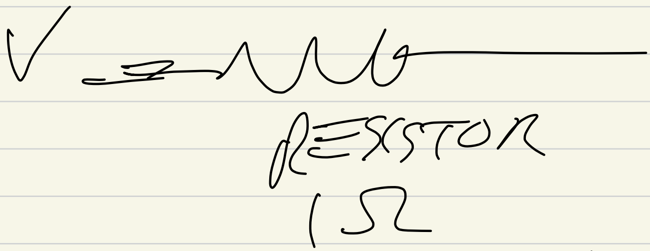
$K = \#$ HEADS.

IF WE KNOW $K \ge 1$, WHAT ARE PROBS FOR $K = 1, 2, 3$?

$$P(K \mid K \ge 1) = \frac{P(K \cap K \ge 1)}{P(K \ge 1)} \quad = \frac{7}{8}$$

K	
0	0
1	$\frac{3/8}{7/8} = 3/7$
2	$3/8 \,/\, 7/8$ $\quad 3/7$
3	$1/8 \,/\, 7/8 = 1/7$

SQUARE LAW DEVICE



V IS R.V.

$$V \in \{-3, -1, 1, 3\} \qquad P = \tfrac{1}{4} \text{ EACH CASE}$$

$$E[V] = \tfrac{1}{4}(-3 + -1 + 1 + 3) = 0$$

$$\text{POWER} = V^2$$

$$E[V^2] = \sum \tfrac{1}{4}\cdot 9 + \tfrac{1}{4}\cdot 1 + \tfrac{1}{4}\cdot 1 + \tfrac{1}{4}\cdot 9$$

$$= \frac{20}{4} = 5$$

PAGE 115-116 IS A USEFUL TABLE OF DISCRETE R.V.