CONDITIONAL EXPECTATIONS

LAST TIME:

$$P_{Y|X}(y|x) \quad \text{CONDITIONAL PMF}$$

$$f_{Y|X}(y|x) \quad \text{CONDITIONAL PDF}$$

NOW WE CAN COMPUTE

DISCRETE:

$$E(Y|x_k) = \sum_{y_j} P_Y(y_j|x_k) \cdot y_j$$

CONTINUOUS:

$$E(Y|x) = \int y\, f_Y(y|x)\, dy$$

BOTH OF THESE ARE FUNCTIONS OF THE GIVEN VALUE $X = x$.

THINK OF THIS AS A FUNCTION $g(x) = E(Y|X)$

NOW WE COULD ASK, WHAT IS THE EXPECTED VALUE OF $g(x)$ WITH RESPECT TO X?

$$E(g(x))$$

$$= E(E(Y|X))$$

$$E(g(x)) = \int g(x) f_X(x)\, dx$$

$$= \int \underbrace{E(Y|x)} f_X(x)\, dx$$

$$= \int \left(\int y f_{Y|X}(y|x)\, dy \right) f_X(x)\, dx$$

$$= \int y \int \underbrace{f_{Y|X}(y|x) f_X(x)}_{\text{CONDITIONAL} \cdot \text{MARGINAL}}\, dx\, dy$$

$$= \int y \left[\int \overset{\text{JOINT}}{f_{XY}(x,y)}\, dx \right] dy$$

$$= \int y f_Y(y)\, dy = E(Y)$$

$$E(E(Y|X)) = E(Y)$$

LAW OF ITERATED EXPECTATIONS.

Ex COIN FLIPPING:

X = # HEADS IN 3 FLIPS

Y = POS 1ST HEAD (0 IF NO HEADS)

JOINT PMF: $P_{XY}(x,y)$

$X \backslash Y$	0 ($\frac{1}{8}$)	1 ($\frac{1}{2}$)	2 ($\frac{1}{4}$)	3 ($\frac{1}{8}$)
0	$\frac{1}{8}$			
1		$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2		$\frac{1}{4}$	$\frac{1}{8}$	
3		$\frac{1}{8}$		

CONDITIONAL PMF OF $Y|X$

	$X \backslash Y$	0	1	2	3	$P_Y(y\|x)$
$\frac{1}{8}$	0	1	0	0	0	$\rightarrow$ ADDS TO 1
$\frac{3}{8}$	1	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\rightarrow$
$\frac{3}{8}$	2	0	$\frac{2}{3}$	$\frac{1}{3}$	0	$\rightarrow$
$\frac{1}{8}$	3	0	1	0	0	$\rightarrow$

$$E(Y|X=0) = 0$$

$$E(Y|X=1) = \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 2 + \frac{1}{3}3 = 2$$

$$E(Y|X=2) = \frac{2}{3}\cdot 1 + \frac{1}{3}\cdot 2 = \frac{4}{3}$$

$$E(Y|X=3) = 1.$$

$$E(Y) = E\big(E(Y|X)\big)$$

$$= 0 \cdot p_X(0) + 2 \cdot p_X(1) + \frac{4}{3} \cdot p_X(2) + 1 \cdot p_X(3)$$

$$= 0 \cdot \frac{1}{8} + 2 \cdot \frac{3}{8} + \frac{4}{3} \cdot \frac{3}{8} + 1 \cdot \frac{1}{8}$$

$$= \frac{6}{8} + \frac{4}{8} + \frac{1}{8} = \boxed{\frac{11}{8}.}$$

DIRECT:

$$0 \cdot \frac{1}{8} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8}$$

$$= \frac{4}{8} + \frac{4}{8} + \frac{3}{8} = \boxed{\frac{11}{8}.}$$

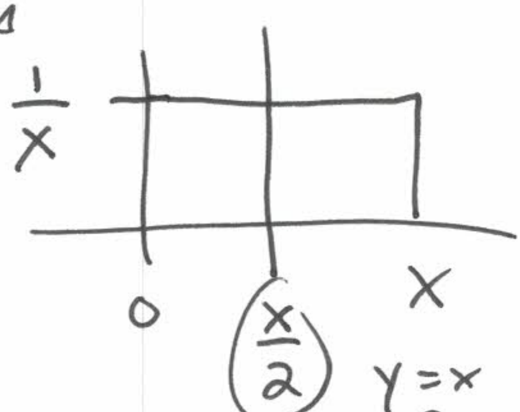
EX CONTINUOUS EXAMPLE FROM LAST TIME:

PICK X UNIFORM ON $[0,1]$

PICK Y UNIFORM ON $[0,X]$.

WHAT IS $E(Y|X)$?

WHAT IS $f_{Y|X}(y|x)$?



$$E(Y|X) = \int_{y=0}^{y=x} y \cdot \frac{1}{x}\, dy$$

$$= \frac{1}{x} \cdot \frac{1}{2} y^2 \Big]_{y=0}^{y=x} = \frac{1}{x} \cdot \frac{1}{2} x^2$$

$$= \frac{x}{2}$$

THEN WHAT IS $E(Y)$?

LAST TIME:

$$f_Y(y) = -\ln y \quad y \in [0,1].$$

$$E(Y) = \int_0^1 -y \ln y \, dy$$

INTEGRATION TABLE OR INTEGRATION BY PARTS

OR

$$E(Y) = E(E(Y|X))$$

$$= E\left(\frac{X}{2}\right) =$$

$$\int_0^1 \frac{x}{2} \cdot 1 \, dx =$$

$$\frac{1}{4} x^2 \Big]_{x=0}^{x=1} = \boxed{\frac{1}{4}}$$

Ex BINARY COMMUNICATIONS CHANNEL

$$X = 1 \text{ or } -1$$

$$Y = X + N$$

GAUSSIAN $\mu = 0$, σ^2

$$f_{Y|X=1} = \text{GAUSSIAN } \mu = 1, \sigma^2$$

$$f_{Y|X=-1} = \text{GAUSSIAN } \mu = -1, \sigma^2$$

$$E(Y) = E(Y|X=1)\,P(X=1) + E(Y|X=-1)\,P(X=-1)$$

$$= 1 \cdot P(X=1) + -1 \cdot P(X=-1)$$

$$= P(X=1) - P(X=-1)$$

FOR REFERENCE:

IF WE HAVE JOINTLY GAUSSIAN r.vs: $\mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho$.

$$\mu_{Y|X} = \boxed{\mu_Y + \frac{(x-\mu_X)\rho\sigma_Y}{\sigma_X}}$$

$$\sigma^2_{Y|X} = \sigma_Y^2(1-\rho^2)$$

(LAST TIME: $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$)

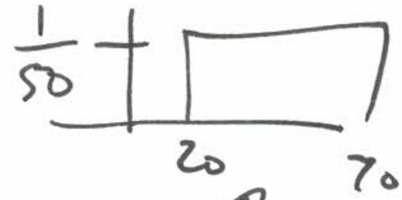
$$\Rightarrow \mu_{Y|X} = \rho x$$

$$\sigma^2_{Y|X} = (1-\rho^2)$$

~~$E(Y|X) =$~~,

$$E(Y) = E(E(Y|X)) = \boxed{\mu_Y} + \cancelto{0.}{\frac{(\mu_X-\mu_X)\rho\sigma_Y}{\sigma_X}}$$

EX LIFE EXPECTANCY.



$X =$ AGE OF A RANDOMLY SELECTED PERSON, UNIFORM ON $[20, 70]$.

$Y =$ YEARS LEFT TO LIVE FOR A PERSON OF AGE X.

EXPONENTIAL WITH $\lambda = \frac{1}{80 - X}$.

$$f_{Y|X} = \lambda e^{-\lambda y} \qquad y \geq 0$$

$$= \frac{1}{80-x} e^{-\frac{y}{80-x}} \qquad y \geq 0, \; x \in [20, 70].$$

$$E(Y|X) = \frac{1}{\lambda} = 80 - X.$$

$$E(Y) = E(E(Y|X))$$

$$= \int_{x=20}^{70} (80-x)\frac{1}{50}\,dx$$

$$= \frac{1}{50}\left(80x - \frac{1}{2}x^2\right]_{x=20}^{x=70}\Big)$$

$$= 35.$$

EX T = TIME REQUIRED FOR CUSTOMER SERVICE

EXPONENTIAL WITH PARAM α.

N = # CUSTOMERS THAT ARRIVE FOR SERVICE ~~AT~~ DURING INTERVAL OF LENGTH T.

POISSON WITH PARAMETER βT.

$$f_T(T) = \alpha e^{-\alpha t} \qquad t > 0.$$

$$E(T) = \frac{1}{\alpha}, \quad Var(T) = \frac{1}{\alpha^2}$$

$$P_{N|T}(n|t) = \frac{(\beta t)^k}{k!} e^{-\beta t} \qquad n = 0, 1, 2, \ldots$$

$$E(N|T) = \beta T$$

$$Var(N|T) = \beta T$$

$$E(N^2|T) = E$$

WHAT IS $E(N)$?

$$E(N) = E(E(N \mid T))$$

$$= E(\beta T) = \boxed{\frac{\beta}{\alpha}}$$

$$Var(N) = \underbrace{E(N^2)}_{?} - (E(N))^2$$

$$E(N^2) = E(E(N^2 \mid T))$$

$$= \int_0^\infty \left(\beta T + (\beta T)^2\right) f_T(t)\, dt$$

$$Var(x) = E(x^2) - E(x^{\cancel{2}})^2$$

$$E(x^2) = Var(x) + (E(x))^2$$

$$= \beta E(T) + \beta^2 E(T^2)$$

BACK TO $Var(N)$:

$$= \cancel{E(}$$

$$\underbrace{\beta E(T)} + \beta^2 E(T^2) - \beta^2 (E(T))^2$$

$$= \cancel{Var}\ \frac{\beta}{\alpha} + \beta^2 Var(T)$$

$$= \boxed{\frac{\beta}{\alpha} + \left(\frac{\beta}{\alpha}\right)^2}$$

IDEA: $E(N) = \frac{\beta}{\alpha}$. IF

$Var(T) = 0$. $E(T) = c$

$E(N) = \beta \cdot c$. JUST LIKE POISSON.

$Var(N) = \beta c$.

IF T IS RANDOM, THEN

$$\mathrm{Var}(N) = \frac{\beta}{\alpha} + \beta^2 \, \mathrm{Var}(T)$$

MORE VARIATION IN $T \Rightarrow$

MORE VARIATION IN N.

KEY TAKEAWAY:

$$\boxed{E(\underbrace{h(Y)}_{\text{WHATEVER FUNCTION.}})) = E(E(h(Y) \mid X))}$$