6.4

a) $F_{\bar{X}}(x_1, x_2, x_3) = P[N_1 \le x_1 - s, N_2 \le x_2 - s, N_3 \le x_3 - s]$

$= F_{N_1}(x_1 - s) F_{N_2}(x_2 - s) F_{N_3}(x_3 - s)$

$f_{\bar{X}}(x_1, x_2, x_3) = f_{N_1}(x_1 - s) f_{N_2}(x_2 - s) f_{N_3}(x_3 - s)$

$= \dfrac{e^{-(x_1-s)^2/2}}{\sqrt{2\pi}} \dfrac{e^{-(x_2-s)^2/2}}{\sqrt{2\pi}} \dfrac{e^{-(x_3-s)^2/2}}{\sqrt{2\pi}}$

b) $P[\min(x_1, x_2, x_3) > 0] = P[X_1 > 0] P[X_2 > 0] P[X_3 > 0]$

$F_Y(y) = 1 - (1 - F_{N_1}(-s))(1 - F_{N_2}(-s))(1 - F_{N_3}(-s))$

$= 1 - (1 - F_N(-s))^3$

$= 1 - (1 - \Phi_N(-s))^3$

c) $P[X_1 > 0, X_2 > 0, X_3 > 0] + P[X_1 > 0, X_2 > 0, X_3 \le 0]$
$+ P[X_1 \le 0, X_2 > 0, X_3 > 0] + P[X_1 > 0, X_2 \le 0, X_3 > 0]$

$= (1 - F_N(-s))^3 + 3F_N(-s)(1 - F_N(-s))^2$

6.23

4.54

$$
\begin{aligned}
M &= \frac{1}{2}X_1 + \frac{1}{2}X_2 = \frac{1}{2}(X_1 + X_2) \\
V &= \frac{1}{2}(X_1 - M)^2 + \frac{1}{2}(X_2^2 - M)^2 = \frac{1}{2}\left(\frac{X_1}{2} - \frac{X_2}{2}\right)^2 + \frac{1}{2}\left(\frac{X_2}{2} - \frac{X_1}{2}\right)^2 \\
&= \frac{1}{8}(X_1 - X_2)^2
\end{aligned}
$$

a

$$
\left.\begin{array}{l} \sqrt{8V} = X_1 - X_2 \\ 2M = X_1 + X_2 \end{array}\right\} \quad \begin{array}{l} X_1 = M + \sqrt{2V} \\ X_2 = M - \sqrt{2V} \end{array}
$$

$$
\begin{aligned}
\begin{vmatrix} \frac{\partial M}{\partial x_1} & \frac{\partial M}{\partial x_2} \\ \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{vmatrix} &= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4}(x_1 - x_2) & -\frac{1}{4}(x_1 - x_2) \end{vmatrix} = \left|-\frac{1}{8}(x_1 - x_2) - \frac{1}{8}(x_1 - x_2)\right| \\
&= \frac{1}{4}|x_1 - x_2| = \frac{1}{4}|2\sqrt{2V}| = \sqrt{v/2}
\end{aligned}
$$

$$
f_{M,V}(m, v) = \frac{f_{X_1,X_2}(m + \sqrt{2v}, m - \sqrt{2v})}{\sqrt{v/2}}
$$

6.23 b

$$
f_{X_1 X_2}(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}
$$

$$
f_{MV}(m, v) = \frac{1}{2\pi\sqrt{v/2}} e^{-\frac{1}{2}\left[(m+\sqrt{2v})^2 + (m-\sqrt{2v})^2\right]}
$$

$$
= \frac{e^{-\frac{1}{2}[2m^2 + 4v]}}{2\pi\sqrt{v/2}} = \underbrace{\frac{e^{-m^2}}{\sqrt{2\pi\frac{1}{2}}}}_{\text{Gaussian}} \; \underbrace{\frac{e^{-2v}}{\sqrt{4\pi v}}}_{\text{chi-square}}
$$

(c)

If $f_{X_1X_2}(x,y) = \lambda e^{-\lambda x}\lambda e^{-\lambda y} \qquad x,y > 0$. Then

$$f_{M,V}(m,v) = \frac{\lambda^2 e^{-\lambda(m+\sqrt{2v})}e^{-\lambda(m-\sqrt{2v})}}{\sqrt{v/2}} \qquad \underbrace{\underbrace{m-\sqrt{2v}>0}_{m>\sqrt{2v}>0}}_{\frac{m^2}{2}>v>0}$$

$$= \frac{\lambda^2 e^{-2\lambda m}}{\sqrt{v/2}} \qquad 0 < v < \frac{m^2}{2}$$

As a check, find $f_M(m)$:

$$f_M(m) = \int_0^{m^2/2} \frac{\lambda^2 e^{-2\lambda m}}{\sqrt{v/2}} dv = \lambda^2 e^{-2\lambda m} 2m$$

This is an Erlang RV for the sum of 2 exponential RV's

6.32

$$m_X = E[\bar{X}] = \begin{bmatrix} S \\ S \\ S \end{bmatrix}$$

$$VAR[X_k] = VAR[S + N_k] = 1$$

$$K_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$