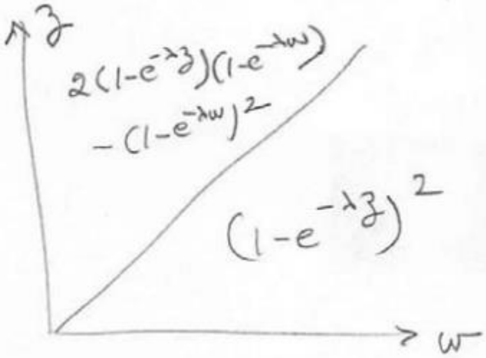
5.103 $F_{XY}(x,y) = (1-e^{-\lambda x})(1-e^{-\lambda y}) \quad x>0 \; y>0$

If $z < w$ then



$$F_{WZ}(w,z) = F_{XY}(z,z)$$
$$= (1-e^{-\lambda z})^2$$

If $z > w$ then from Ex. 5.43

$$F_{WZ}(w,z) = F_{XY}(w,z) + F_{XY}(z,w) - F_{XY}(w,w)$$
$$= 2(1-e^{-\lambda z})(1-e^{-\lambda w}) - (1-e^{-\lambda w})^2$$

Check: find marginal cdf of $W = \min(X,Y)$.

$$\lim_{z\to\infty} F_{WZ}(w,z) = 2(1-e^{-\lambda w}) - (1-e^{-\lambda w})^2$$
$$= 1 - e^{-2\lambda w} \qquad w>0$$

exponential w. rate 2λ

5.107

(a) $$\begin{bmatrix} V \\ W \end{bmatrix} = \begin{bmatrix} a & b \\ c & e \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

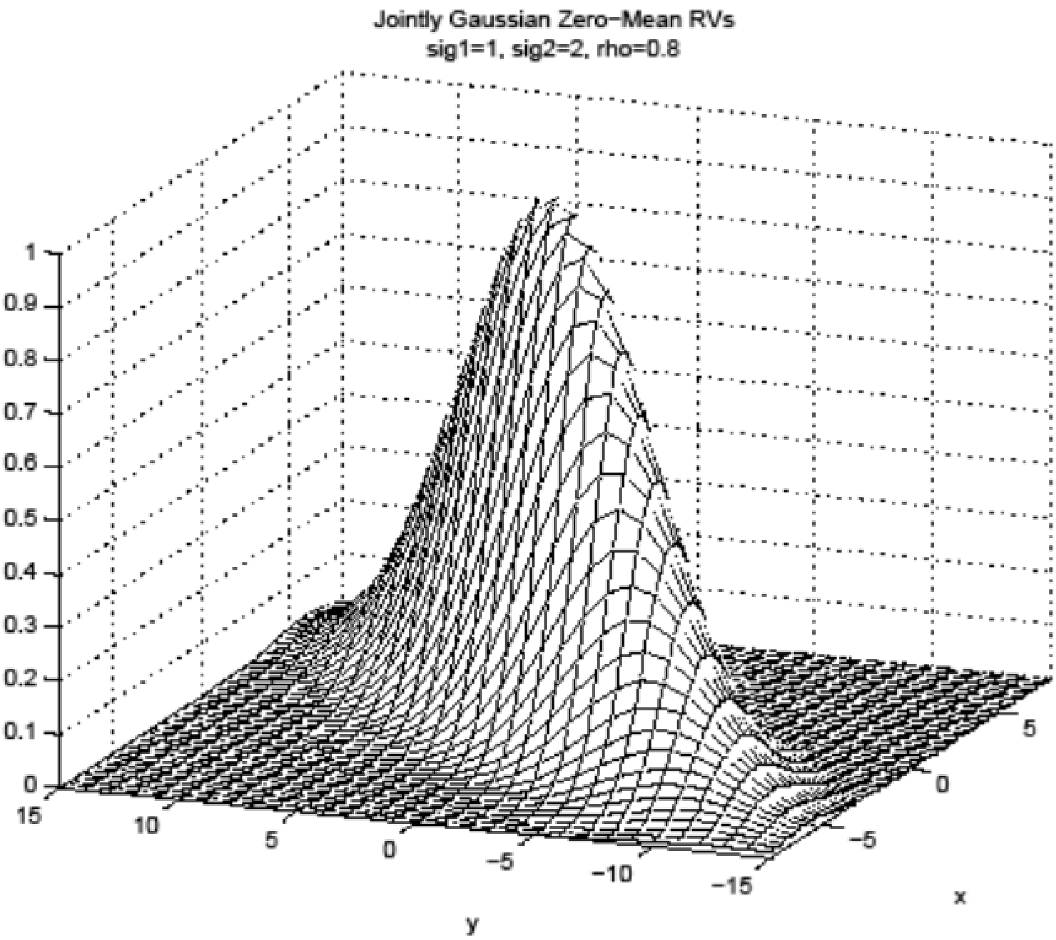
then, $$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{|ae-bc|} \begin{bmatrix} e & -b \\ -c & a \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix}$$

$$f_{VW} = f_{XY}\left(\frac{ev-bw}{|ae-bc|}, \frac{-cv+aw}{|ae-bc|}\right)$$

$$= f_X\left(\frac{ev-bw}{|ae-bc|}\right) f_Y\left(\frac{-cv+aw}{|ae-bc|}\right)$$

where f_X and f_Y Gaussian pdf with $\mu = 0$ and $\sigma = 1$.

(e)


Jointly Gaussian Zero−Mean RVs
sig1=1, sig2=2, rho=0.8
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
15
10
5
0
−5
−10
−15
y
5
0
−5
x

5.131 a)

$$
\begin{aligned}
f_{X,R}(x,r) &= f_X(x|r)f_R(r) \\
&= re^{-rx}\frac{\lambda(\lambda r)^{\alpha-1}e^{-\lambda r}}{\Gamma(\alpha)}
\end{aligned}
$$

b)

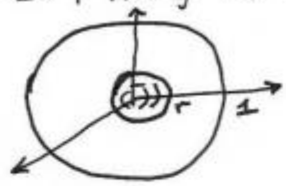
$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,R}(x,r)dr \\
&= \int_0^{\infty} \frac{(\lambda r)^{\alpha}e^{-(\lambda+X)r}}{\Gamma(\alpha)}dr \\
&= \int_0^{\infty} \frac{\lambda^{\alpha}}{(\lambda+x)^{\alpha}}\frac{[(\lambda+X)r]^{\alpha}e^{-(\lambda+X)r}}{\Gamma(\alpha)}dr \\
&= \frac{\lambda^{\alpha}}{(\lambda+X)^{\alpha}\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha+1)}{\lambda+X} \\
&= \frac{\alpha\lambda^{\alpha}}{(\lambda+x)^{\alpha+1}} \qquad x>0
\end{aligned}
$$

c)

$$
\begin{aligned}
E[X] &= \int_0^{\infty} xf_X(x)dx \\
&= \int_0^{\infty} \frac{\alpha\lambda^{\alpha}x}{(\lambda+x)^{\alpha+1}}dx \\
&= \frac{\lambda^{\alpha}}{-\alpha+1}(\lambda+x)^{-\alpha+1}\Big|_0^{\infty} \\
&= \frac{\lambda}{\alpha-1} \qquad (\alpha>1) \\
E[X^2] &= \int_0^{\infty} x^2 f_X(x)dx \\
&= \int_0^{\infty} \frac{\alpha\lambda^{\alpha}x^2}{(\lambda+x)^{\alpha+1}}dx \\
&= -\lambda^{\alpha}\int_0^{\infty} x^2 d(\lambda+x)^{-\alpha} \\
&= -\lambda^{\alpha}x^2(\lambda+x)^{-\alpha}\Big|_0^{\infty} + \lambda^{\alpha}\int_0^{\infty}(\lambda+x)^{-\alpha}2xdx \\
&= \frac{2\lambda^{\alpha}}{-\alpha+1}\int_0^{\infty} xd(\lambda+x)^{-\alpha+1} \\
&= \frac{2\lambda^{\alpha}}{-\alpha+1}x(\lambda+x)^{-\alpha+1}\Big|_0^{\infty} + \frac{2\lambda^{\alpha}}{\alpha-1}\int_0^{\infty}(\lambda+x)^{-\alpha+1}dx \\
&= \frac{2\lambda^{\alpha}}{-\alpha-1}\cdot\frac{1}{-\alpha+2}(\lambda+x)^{-\alpha+2}\Big|_0^{\infty} \\
&= \frac{2\lambda^2}{(\alpha-1)(\alpha-2)} \\
VAR[X] &= E[X^2]-E^2[X] = \frac{2\lambda^2}{(\alpha-1)(\alpha-2)} - \frac{\lambda^2}{(\alpha-1)^2}
\end{aligned}
$$

6.1

a) $P[X^2+Y^2+Z^2 \le r \mid X^2+Y^2+Z^2 \le 1] = \frac{4\pi r^3}{3} \frac{3}{4\pi 1^3} = r^3, \; r \le 1$



b) $P[|X| \le \frac{1}{\sqrt{3}} \cap |Y| < \frac{1}{\sqrt{3}} \cap |Z| \le \frac{1}{\sqrt{3}} \mid X^2+Y^2+Z^2 \le 1]$

$$= \frac{\frac{2}{\sqrt{3}} \cdot \frac{2}{\sqrt{3}} \cdot \frac{2}{\sqrt{3}}}{\frac{4\pi 1^3}{3}} = \frac{8}{3\sqrt{3}4\pi} = \frac{2}{\pi\sqrt{3}} = 0.3675$$

c) $P[X > 0 \cap Y > 0 \cap Z > 0] = \frac{1}{8}$

d) $P[Z \le 0 \mid X^2+Y^2+Z^2 \le 1] = \frac{1}{2}$

6.7 a)

$$\begin{aligned} 1 &= \int_0^1 \int_0^1 \int_0^1 k(x+y+z)dxdydz \\ &= k\int_0^1 \int_0^1 \left(\frac{1}{2}+y+z\right) dydz \\ &= k\int_0^1 \left(\left(\frac{1}{2}+z\right)+\frac{1}{2}\right) dz \\ &= k\left(1+\frac{1}{2}\right) \Rightarrow k = \frac{2}{3} \end{aligned}$$

b) $f_{XY}(x,y) = \frac{2}{3}\int_0^1 (x+y+z)dz = \frac{2}{3}\left[x+y+\frac{1}{2}\right]$

$$f_Z(z|x,y) = \frac{f_{XYZ}(x,y,z)}{f_{XY}(x,y)} = \frac{x+y+z}{x+y+\frac{1}{2}}$$

c) $f_X(x) = \frac{2}{3}\int_0^1 (x+y+\frac{1}{2})\,dy = \frac{2}{3}\left[x\,y\Big|_0^1 + \frac{y^2}{2}\Big|_0^1 + \frac{1}{2}y\Big|_0^1\right] = \frac{2}{3}\left[x+1\right]$

6.14

a) $$p(X_1=x_1, X_2=x_2, X_3=x_3, X_4=x_4) = \frac{n!}{x_1!\,x_2!\,x_3!\,x_4!}\left(\tfrac{1}{2}\right)^{x_1}\left(\tfrac{1}{4}\right)^{x_2}\left(\tfrac{1}{8}\right)^{x_3}\left(\tfrac{1}{8}\right)^{x_4}$$

$$p(X_1=x_1, X_2=x_2, X_3=x_3) = \frac{n!\,\left(\tfrac{1}{2}\right)^{x_1}\left(\tfrac{1}{4}\right)^{x_2}\left(\tfrac{1}{8}\right)^{x_3}\left(\tfrac{1}{8}\right)^{n-x_1-x_2-x_3}}{x_1!\,x_2!\,x_3!\,(n-x_1-x_2-x_3)!}$$

b) $$p(x_1,x_2) = \sum_{x_3=0}^{n} \frac{n!\,\left(\tfrac{1}{2}\right)^{x_1}\left(\tfrac{1}{4}\right)^{x_2}\left(\tfrac{1}{8}\right)^{x_3}\left(\tfrac{1}{8}\right)^{n-x_1-x_2-x_3}}{x_1!\,x_2!\,x_3!\,(n-x_1-x_2-x_3)!}$$

$$= \frac{\left(\tfrac{1}{2}\right)^{x_1}\left(\tfrac{1}{4}\right)^{x_2}}{x_1!\,x_2!} \sum_{x_3=0}^{n} \frac{(n-x_1-x_2)!}{x_3!\,(n-x_1-x_2-x_3)!}\,\frac{n!}{(n-x_1-x_2)!}\left(\tfrac{1}{8}\right)^{x_3}\left(\tfrac{1}{8}\right)^{n-x_1-x_2-x_3}$$

$$= \frac{\left(\tfrac{1}{2}\right)^{x_1}\left(\tfrac{1}{4}\right)^{x_2}}{x_1!\,x_2!}\,\frac{n!}{(n-x_1-x_2)!}\left(\frac{1}{8}+\frac{1}{8}\right)^{n-x_1-x_2}$$

c) $$p(x_1) = \sum_{x_2=0}^{n} \left(\tfrac{1}{2}\right)^{x_1}\left(\tfrac{1}{4}\right)^{x_2}\left(\tfrac{1}{4}\right)^{n-x_1-x_2}\frac{n!}{x_1!\,x_2!\,(n-x_1-x_2)!}$$

$$= \frac{\left(\tfrac{1}{2}\right)^{x_1}}{x_1!}\sum_{x_2=0}^{n}\frac{n!}{(n-x_1)!}\cdot\frac{(n-x_1)!}{(n-x_1-x_2)!\,x_2!}\left(\tfrac{1}{4}\right)^{x_2}\left(\tfrac{1}{4}\right)^{n-x_1-x_2}$$

$$= \frac{\left(\tfrac{1}{2}\right)^{x_1}}{x_1!}\,\frac{n!}{(n-x_1)!}\left(\tfrac{1}{2}\right)^{n-x_1}$$

$$= \binom{n}{x_1}\left(\tfrac{1}{2}\right)^{n}$$

d) $$p(x_2,x_3 \mid x_1=m) = \frac{\frac{n!}{m!\,x_2!\,x_3!\,(n-x_2-x_3-m)!}\left(\tfrac{1}{2}\right)^{m}\left(\tfrac{1}{4}\right)^{x_2}\left(\tfrac{1}{8}\right)^{x_3}\left(\tfrac{1}{8}\right)^{n-m-x_2-x_3}}{\frac{n!}{m!\,(n-m)!}\left(\tfrac{1}{2}\right)^{n}}$$

$$= \frac{(n-m)!}{x_2!\,x_3!\,(n-m-x_2-x_3)!}\underbrace{\left(\tfrac{1}{2}\right)^{m}\left(\tfrac{1}{2}\right)^{-n}\left(\tfrac{1}{4}\right)^{x_2}\left(\tfrac{1}{8}\right)^{x_3}\left(\tfrac{1}{8}\right)^{n-m-x_2-x_3}}_{\left(\frac{2}{4}\right)^{x_2}\left(\frac{2}{8}\right)^{x_3}\left(\frac{2}{8}\right)^{n-m-x_2-x_3}}$$