5.3

(a) Sample Space : a set of outcomes where each outcome is a pair $\underline{\zeta} = (\zeta_1, \zeta_2)$ where ζ_1 is the input and ζ_2 is the output.

$$S_{XY} = \{(-1,-1), (-1,0), (-1,1), (1,-1), (1,0), (1,1)\}$$

(b) $P[X=1, Y=-1]$

$= P[Y=-1 \mid X=1] P[X=1]$

$= \frac{3}{4} p$

$P[X=-1, Y=-1]$

$= \frac{1}{4}(1-p-p_e)$

$P[X=1, Y=0]$

$= P[Y=0 \mid X=1] P[X=1]$

$= \frac{3}{4} p_e$

$P[X=-1, Y=0]$

$= \frac{1}{4} p_e$

$P[X=1, Y=1]$

$= P[Y=1 \mid X=1] P[X=1]$

$= \frac{3}{4}(1-p-p_e)$

$P[X=-1, Y=1]$

$= \frac{1}{4} p$

(c) $P[X \neq Y]$

$= \frac{1}{4} p_e + \frac{1}{4} p + \frac{3}{4} p_e + \frac{3}{4} p$

$= p_e + p$

$P[Y=0]$

$= \frac{3}{4} p_e + \frac{1}{4} p_e$

$= p_e$

5.11

(i)

X \ Y	−1	0	1	
−1	1/6	1/6	0	1/3
0	0	0	1/3	1/3
1	1/6	1/6	0	1/3
	1/3	1/3	1/3	

$P[X=i] = \frac{1}{3} \quad i \in \{-1,0,1\}$

$P[Y=i] = \frac{1}{3} \quad i \in \{-1,0,1\}$

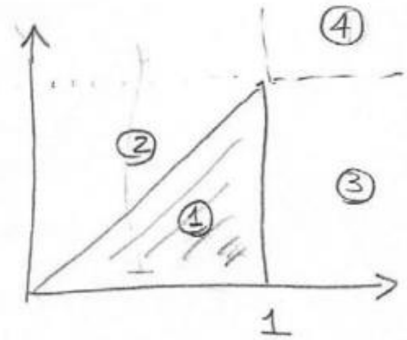
$P[X>0] = \frac{1}{3}$

$P[X \geq Y] = \frac{1}{2}$

$P[X=-Y] = 1/6$

5.17

(a)

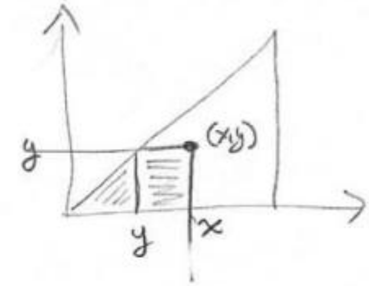


Area of Triangle is 1/2

Region 1

$0 < y < x < 1$

$$P[X \le x, Y \le y] = \frac{\overbrace{\frac{y^2}{2}}^{\text{triangle}} + \overbrace{y(x-y)}^{\text{rectangle}}}{1/2}$$

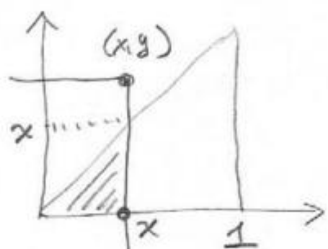


$$= 2\left(xy - \frac{y^2}{2}\right)$$

Region 2

$0 < x < y$

$$P[X \le x, Y \le y] = \frac{x^2/2}{1/2} = x^2$$

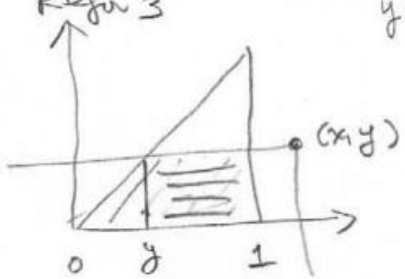


Region 3

$y < x,\ x > 1$

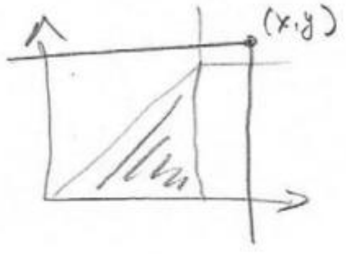
$$P[X \le x, Y \le y] = \frac{\frac{y^2}{2} + y(1-y)}{1/2}$$

$$= 2\left(y - \frac{y^2}{2}\right)$$

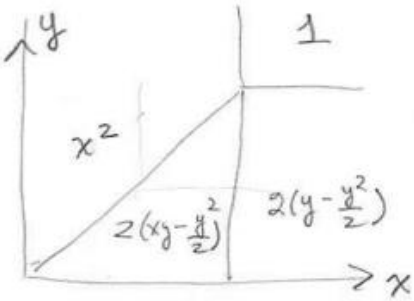


$x > 1,\ y > 1$

$$P[X \le x, Y \le y] = 1$$



5.17



(b) $P[X \le x] = F_{XY}(x, \infty) = x^2$

$P[Y \le y] = F_{XY}(\infty, y) = 2\left(y - \frac{y^2}{2}\right)$

(c) $P\left[X \le \frac{1}{2}, Y \le \frac{3}{4}\right] = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$ since $\left(\frac{1}{2}, \frac{3}{4}\right)$ in Region 2

$$P\left[\frac{1}{4} < X \le \frac{3}{4}, \frac{1}{4} < Y \le \frac{3}{4}\right]$$

$$= F_{XY}\left(\frac{3}{4}, \frac{3}{4}\right) - F_{XY}\left(\frac{3}{4}, \frac{1}{4}\right) - F_{XY}\left(\frac{1}{4}, \frac{3}{4}\right) + F_{XY}\left(\frac{1}{4}, \frac{1}{4}\right)$$

$$= \left(\frac{3}{4}\right)^2 - 2\left(\frac{3}{4}\cdot\frac{1}{4} - \frac{1}{2}\left(\frac{1}{4}\right)^2\right) - \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2$$

$$= \frac{1}{4}$$

5.25 (a) for $x>0, y>0$

$$F_{XY}(x,y) = \int_0^x \int_0^y \tfrac{1}{2} e^{-x/2}\, 2y e^{-y^2}\, dx\, dy$$

$$= (1-e^{-x/2})(1-e^{-y^2})$$

(b)

$$P[X>Y] = \int_0^\infty \int_0^x 2y e^{-y^2} dy\, \tfrac{1}{2} e^{-x/2} dx$$

$$= \int_0^\infty (1-e^{-x^2}) \tfrac{1}{2} e^{-x/2} dx \qquad (x+\tfrac{1}{4})^2 = \tfrac{1}{16}$$

$$= 1 - \tfrac{1}{2}\int_0^\infty e^{-(x^2+x/2)} dx \qquad = x^2 + \tfrac{1}{2}x + \tfrac{1}{16} - \tfrac{1}{16}$$

$$= 1 - \tfrac{1}{2} e^{1/16} \underbrace{\int_0^\infty e^{-(x+\frac{1}{4})^2} dx}_{\sqrt{2\pi \frac{1}{2}} = \sqrt{\pi}}$$

← Gaussian pdf mean $-1/4$ variance $1/2$

$$= 1 - \frac{\sqrt{\pi}\, e^{1/16}}{2}$$

(c)

$$F_X(x) = \lim_{y\to\infty} F_{XY}(x,y) = 1-e^{-x/2} \qquad x>0$$

$$f_X(x) = \tfrac{1}{2} e^{-x/2}$$

$$F_Y(y) = 1-e^{-y^2}$$

$$f_Y(y) = 2y\, e^{-y^2}$$

5.41

Let M represent Michael's arrival time (minutes after 7:00)
Let B represent the arrival time of the bus (minutes after 7:00)

M, B uniform RVs

$f_M(m) = \frac{1}{15} \quad , 25 \le m \le 40$

$f_B(b) = \frac{1}{10} \quad , 27 \le b \le 37$

(a) $f_{MB}(m,b) = f_M(m) f_B(b)$ since M, B independent

$= \frac{1}{150} \quad , 25 \le m \le 40, \ 27 \le b \le 37$

$0 \quad$, otherwise

$$P[M+5 < B] = \int_{30}^{37} \int_{25}^{b-5} \left(\frac{1}{150}\right) dm\,db$$

$$= \frac{1}{150} \int_{30}^{37} (b-30)\,db$$

$$= \frac{1}{150} \left[\frac{b^2}{2} - 30b \right]_{30}^{37}$$

$$= 0.163$$

(b) $$P[M > B] = \int_{27}^{37} \int_{b}^{40} \frac{1}{150}\, dm\,db$$

$$= \frac{1}{150} \int_{27}^{37} (40-b)\,db$$

$$= \frac{1}{150} \left[40b - \frac{b^2}{2} \right]_{27}^{37}$$

$$= 8/15$$

$$\cong 0.533$$

5.56

~~4.59~~ a) $\mathcal{E}[(X+Y)^2] = \mathcal{E}[X^2 + 2XY + Y^2] = \mathcal{E}[X^2] + 2\mathcal{E}[XY] + \mathcal{E}[Y^2]$

b)
$$\begin{aligned} VAR[X+Y] &= \mathcal{E}[(X+Y)^2] - \mathcal{E}[X+Y]^2 \\ &= \mathcal{E}[X^2] + 2\mathcal{E}[XY] + \mathcal{E}[Y^2] - \mathcal{E}[X]^2 \\ &\quad -2\mathcal{E}[X]\mathcal{E}[Y] - \mathcal{E}[Y]^2 \\ &= VAR[X] + VAR[Y] + 2[\mathcal{E}[XY] - \mathcal{E}[X]\mathcal{E}[Y]] \end{aligned}$$

c) $VAR[X+Y] = VAR[X] + VAR[Y]$ if $\mathcal{E}[XY] = \mathcal{E}[X]\mathcal{E}[Y]$ that is, if X and Y are uncorrelated.