1. (5 points) This is a followup on last week's first question, which was this:

With a lot of tests, the results are grey, and the person running them has a choice in how to interpret them: lean towards finding someone guilty (but falsely accusing an innocent person), or the other way toward finding someone innocent (but letting a guilty person go free).

Assume that in this example, the administrator can choose the bias. However the sum of the two types of errors is constant at 0.2%. (Whether that relation is really true would depend on the test.)

This question is to plot both the number of innocent people falsely found guilty and the number of guilty people wrongly let go, as a function of the false positive rate. Use any plot package. Both numbers of people will usually be fractional.

$$P[B_c] = \frac{1}{100}$$

$$P[B_c^c] = \frac{99}{100}$$

$$P[\{false\ positive\}] = P[Pos\,|B_c^c] = x$$

$$P[\{true\ positive\}] = P[Pos\,|B_c] = 1 - x$$

$$P[\{false\ negative\}] = P[Neg\,|B_c] = 0.002 - x$$

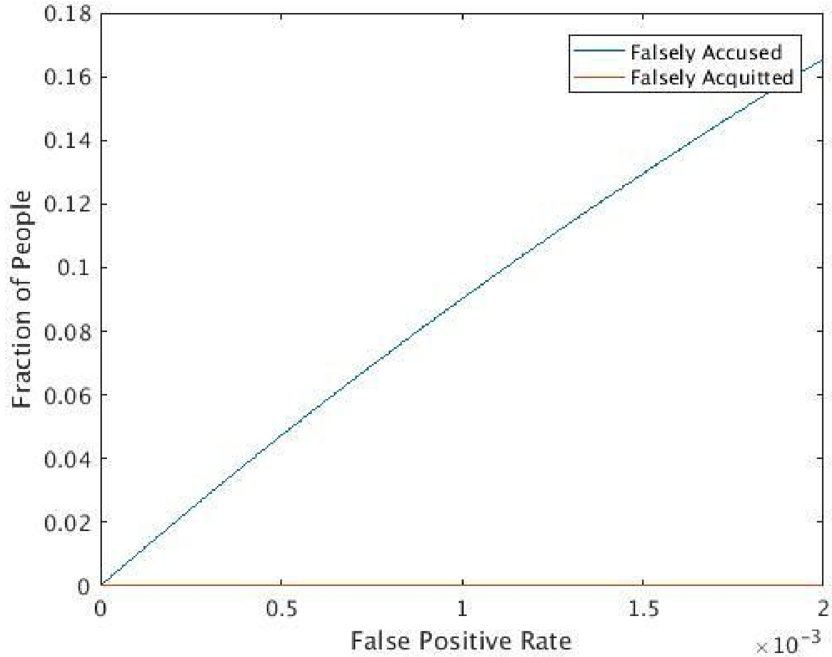
$$P[\{true\ negative\}] = P[Neg\,|B_c^c] = 1 - (0.002 - x)$$

Fraction of people falsely accused:

$$P[\{person\ tested\ pos\ is\ innocent\}] = P[B_c^c \mid Pos] = \frac{P[B_c^c]P[Pos\,|B_c^c]}{(P[Pos\,|B_c^c]P[B_c^c]+P[Pos\,|B_c]P[B_c])}$$

$$P[\{person\ tested\ neg\ is\ guilty\}] = P[B_c \mid Neg] = \frac{P[B_c]P[Neg \mid B_c]}{(P[Neg\,|B_c^c]P[B_c^c]+P[Neg\,|B_c]P[B_c])}$$

Plot:



Code:

```
% A Priori Probabilties
P_C    = 1/100;                 % Criminal
P_NC = 99/100;                  % Not Criminal

% A Posteriori Probabilties
P_FP  = linspace(0,0.02,500);       % False Positive
P_TP  = 1-P_FP;                 % True Positive
P_FN  = 0.002-P_FP;             % False Neg
P_TN  = 1-P_FN;                 % True Neg

% Baye's Rule
Num_FalselyAccused   = (P_NC*P_FP)./(P_TP*P_C+P_FP*P_NC); % P[Not Criminal |
Pos]
```

```
Num_FalselyAcquitted = (P_C*P_FN)./(P_TN*P_NC+P_FN*P_C);  % P[Criminal | Neg]

figure
plot(P_FP, Num_FalselyAccused);
hold on
plot(P_FP, Num_FalselyAcquitted, '-');
legend('Falsely Accused', 'Falsely Acquitted');
xlabel('False Positive Rate');
ylabel('Fraction of People');
```

2.

$P[\text{both in error}] = q_1 q_2$

ⓐ $P[k \text{ transmissions needed}] = (q_1 q_2)^{k-1}(1-q_1 q_2) \quad k=1,2,\ldots$

$P[\text{more than } k \text{ transmissions required}]$

$$= \sum_{j=k+1}^{\infty} (q_1 q_2)^{j-1}(1-q_1 q_2) = (q_1 q_2)^k \sum_{j=0}^{\infty} (1-q_1 q_2)(q_1 q_2)^j$$

$$= (q_1 q_2)^k$$

ⓑ $P[\text{link 2 errorfree} \mid \text{one or more errorfree}]$

$$= \frac{P[\text{one or more errorfree, link 2 errorfree}]}{1-q_1 q_2}$$

$$= \frac{q_1(1-q_2) + (1-q_1)(1-q_2)}{1-q_1 q_2} = \frac{1-q_2}{1-q_1 q_2}$$

3.

(a)

$$p_b = p[N \geq 7] = p[N = 7] + p[N = 8]$$
$$= 7p^7(1-p) + p^8$$

(b)

$$p[N_b \geq 1] = 1 - p[N_b = 0] = 1 - (1 - p_b)^n = 0.999$$

$$0.001 = (1 - p_b)^n$$

$$n = \frac{\log 0.001}{\log(1 - p_b)} = \frac{\log 0.001}{\log(1 - (7(1-p)^7 p + (1-p)^8))}$$

4.

Sample Space:

	Michael \ Carlos	0	1	2
1/4	0	(0,0)	(0,1)	(0,2)
1/2	1	(1,0)	(1,1)	(1,2)
1/4	2	(2,0)	(2,1)	(2,2)
		1/4	1/2	1/4

Probabilities

	0	1	2
0	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
2	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

Mapping $S \to S_X$

	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$P[X=0] = P[(0,0)] = \frac{1}{16}$

$P[X=1] = P[\{(1,0),(1,1),(0,1)\}] = \frac{1}{2}$

$P[X=2] = 3 \times \frac{1}{16} + 2 \times \frac{1}{8} = \frac{7}{16}.$

5.

Let A_i = Transmitter #1 sends a signal at time slot i

B_i = " #2 "

A signal gets through if $A_i B_i^c \cup A_i^c B_i$ occurs

Each experiment has 4 outcomes

(a) Experiment i

	A_i	A_i^c
B_i	1/4	1/4
B_i^c	1/4	1/4

Sample Space consists of a Cartesian product of the outcomes of the basic experiment

$S = (s_1, s_2, \ldots)$ where s_i is an outcome from basic experiment

(b) $X(S) = n$

if n is the first occurrence of $A_i B_i^c \cup A_i^c B_i$ in $s_1, s_2, \ldots$

(c) $P[A_i B_i^c \cup A_i^c B_i] = P[A_i B_i^c] + P[A_i^c B_c] = \frac{1}{2} = P[\text{success}]$

$P[X=k] = P[(k-1)\text{ failures, } 1 \text{ success}] = \left(\frac{1}{2}\right)^k$

6. (a)

$$\sum_{k=1}^{\infty} \frac{c}{k^2} = 1 \to c\frac{\pi^2}{6} = 1 \to c = \frac{6}{\pi^2}$$

(b)

$$P(X > 4) = 1 - P(X \leq 4) = 1 - \frac{6}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}\right) = 0.135$$

(c)

$$P(6 \leq X \leq 8) = \frac{6}{\pi^2}\left(\frac{1}{36} + \frac{1}{49} + \frac{1}{64}\right) = 0.039$$

7.

		Terminal 2 p	$1-p$
Terminal 1	$\frac{1}{2}$	$\frac{1}{2}p$	$\frac{1}{2}q$
	$\frac{1}{2}$	$\frac{1}{2}p$	$\frac{1}{2}q$

$P_{success} = \frac{1}{2}q + \frac{1}{2}p = \frac{1}{2}$ same

$\therefore$ The pmf of X is unchanged.

$$P[\text{Terminal 2 transmitted} \mid \text{success}] = \frac{P[\text{success and Terminal 2 transmitted}]}{P[\text{success}]}$$

$$= \frac{\frac{1}{2}p}{\frac{1}{2}} = p$$

This suggests that terminal 2 should always transmit (at the expense of terminal 1).