Engineering Probability HW 3

1. (5 pts) Assume that it is known that one person in a group of 100 committed a crime. You're in the group, so there's a prior probability of 1/100 that you are it. There is a pretty good forensic test. It makes errors (either way) only 0.1% of the time. You are given the test; the result is positive. Using this positive test, what's the probability now that you are the criminal? (Use Bayes.)

Answers:

P[yes|positive]= $\frac{P[yes \cap positive]}{P[positive]}$ = $\frac{\frac{1}{100} \times 99.9\%}{\frac{1}{100} \times 99.9\% + \frac{99}{100} \times 0.1\%}$ $\approx 91.0\%$

2. (5 pts) Find the probability that in a class of 28 students exactly four were born in each of the seven days of the week. However, make it 21 students and 3 on each day of the week. Assume that there is no relation between birthday and day of the week.

Answers:

P[3 on each day of the work]=21 C_4 $(\frac{1}{7})^3$ $(\frac{6}{7})^{18}$ $\times$ 18 C_4 $(\frac{1}{6})^3$ $(\frac{5}{6})^{15}$ $\times$ 15 C_4 $(\frac{1}{5})^3$ $(\frac{4}{5})^{12}$ $\times$ 12 C_4 $(\frac{1}{4})^3$ $(\frac{3}{4})^9$ $\times$ 9 C_4 $(\frac{1}{3})^3$ $(\frac{2}{3})^6$ $\times$ 6 C_4 $(\frac{1}{2})^3$ $(\frac{1}{2})^3$ $\times$ 3 C_4 $(1)^3$ =0.00327

3. (5 pts) A die is tossed twice and the number of dots facing up is counted and noted in the order of occurrence. Let A be the event "number of dots in rst toss is not less than number of dots in second toss," and let B be the event "number of dots in rst toss is 6." Find P[A|B] and P[B|A].

Answers:

The sample space in event A={(1,1),(2,1),(2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(4,4),(5,1),(5,2),(5,3),(5,4),(5,5),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

The sample space in event B={(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

P[A|B]= $\frac{P[A \cap B]}{P[B]}$ =1

P[B|A]= $\frac{P[A \cap B]}{P[A]}$ = $\frac{2}{7}$

4. (5 pts) A number x is selected at random in the interval [-2, 2]. Let the events A = {x < 0},B = {|x-0.5| <0.5}, and C = {x > 0.75}. Find P[A|B], P[B|C], P[A| C^c], P[B| C^c].

Answers:

A=[-2,0)

B=(0,1)

C=(0.75,2]

P[A|B]=0

P[B|C]= $\frac{P[B \cap C]}{P[C]}$ = $\frac{(0.75,1)}{(0.75,2]}$ =0.2

P[A| C^c]= $\frac{P[A \cap C^c]}{P[C^c]}$ = $\frac{[-2,0)}{[-2,0.75]}$ =0.727
P[B| C^c }= $\frac{P[B \cap C^c]}{P[C^c]}$ = $\frac{(0,0.75]}{[-2,0.75]}$ =0.272

5. (5 pts) A cryptographic hash takes a message as input and produces a xed-length string as output, called the digital fingerprint. A brute force attack involves computing the hash for a large number of messages until a pair of distinct messages with the same hash is found. Find the number of attempts required so that the probability of obtaining a match is 0.5 How many attempts are required to find a matching pair if the digital fingerprint is 64 bits long? 128 bits long?

<u>Answers:</u>
1-P[distinct hash in N trials]=P[same hash in N trials]=0.5
P[distinct hash in N trials]= $\frac{2^L}{2^L} \times \frac{2^L-1}{2^L} \times \frac{2^L-2}{2^L} \times ... \times \frac{2^L-n}{2^L}$ =0.5
n= $2^{\frac{L}{2}}$
L =64, n= 2^{32}
L=128, n= 2^{64}

6. (5 pts) In each lot of 100 items, two items are tested, and the lot is rejected if either of the tested items is found defective.
(a) Find the probability that a lot with k defective items is accepted.
(b) Suppose that when the production process malfunctions, 50 out of 100 items are defective.In order to identify when the process is malfunctioning, how many items should be tested so that the probability that one or more items are found defective is at least 99%?

<u>Answers:</u>
(a)P[k defective]= $\frac{\binom{100-k}{2}}{\binom{100}{2}}$ = $\frac{(100-k)!}{2!(100-k-2)!} \times \frac{2!(100-2)!}{100!}$ = $\frac{(100-k)!}{(98-k)!} \times \frac{1}{100 \times 99}$ = $\frac{(k-99)(k-100)}{100 \times 99}$
(b)P[X $\geq$ 1] $\geq$ 0.99, 1-P[X $\leq$ 1] $\geq$ 0.99
1-P[X=0] $\geq$ 0.99, according to the Binomial formula simply as: 1- 0.5^n $\geq$ 0.99
n $\geq$ 6.67 $\approx$ 7

7. (5 pts) Let S = {1,2,3,4} and A ={1,2}, B ={1,3}, C = {1,4}. Assume the outcomes are equip-probable. Are A, B, and C independent events?

<u>Answers:</u>
P[A $\cap$ B $\cap$ C]=P[{1}]= $\frac{1}{4}$
P[A]=P[B]=P[C]= $\frac{1}{2}$
P[A] $\times$ P[B] $\times$ P[C]= $\frac{1}{8}$ $\neq$ P[A $\cap$ B $\cap$ C]

A B C are not independent events.

8. (5 pts) A block of 100 bits is transmitted over a binary communication channel with probability of bit error p = 10^{-2}

(a) If the block has 1 or fewer errors then the receiver accepts the block. Find the probability that the block is accepted.

(b) If the block has more than 1 error, then the block is retransmitted. Find the probability that M retransmissions are required.

Answers:

(a)P[x=1]= $\binom{100}{1} \times \frac{1}{100} \times (\frac{99}{100})^{99}$ =0.370

P[x=0]= $(\frac{99}{100})^{100}$ =0.366

P[the block is accpeted]=P[x=1]+P[x=0]=0.736

(b)P[send once and need retransmisson]=1-P[x≤ 1]=0.264

According to Geometric Random Variable, P[need M retransmisson]= $0.264^{M-1} \times$ 0.736

9. (5 pts) A machine makes errors in a certain operation with probability p.There are two types of errors.The fraction of errors that are type 1 is α and type 2 is 1-α.

(a) What is the probability of k errors in n operations?

(b) What is the probability of k1 type 1 errors in n operations?

(c) What is the probability of k2 type 2 errors in n operations?

(d) What is the joint probability of k1 and k2 type 1 and 2 errors, respectively, in n operations?

Answers:

(a) C(n,k)$\times p^k \times (1-p)^{n-k}$

(b) C(n,k1)$\times (\alpha p)^{k1} \times (1-\alpha p)^{n-k1}$

(c) C(n,k2)$\times ((1-\alpha)p)^{k2} \times ((1-p)+(\alpha p)^{n-k2}$

(d) C(n,k1)$\times (\alpha p)^{k1} \times$C(n-k1,k2)$\times ((1-\alpha)p)^{k2} \times (1-p)^{n-k1-k2}$

10. (5 pts) A biased coin is tossed repeatedly until heads has come up three times. Find the probability that k tosses are required.

Answers:

A={kth toss is head}
B={2 heads occurs in k -1 tosses}
P[A]=p, P[B]=C(k-1,2) $\times p^2 \times (1-p)^{k-1-2}$
P[A∩B]=P[A] $\times$ P[B]= $\frac{(k-1)(k-2)p^3(1-p)^{k-3}}{2}$